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THEORY OF DISTURBANCE-UTILIZING CONTROL WITH APPLICATION TO MISSILE INTERCEPT PROBLEMS

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12 December 1979

U.S. ARMY MISSILE COMMAND

Redstone Arsenal, Alabama 35809

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Control problems with disturbances which have waveform structure may be addressed by considering the maximum utilization mode of disturbance-accommodating control theory.

It is shown in this work that the optimal utilization of disturbances can lead to significantly better performance than that of the "optimal" linear-quadratic regulator. The conditions
(Continued)

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20. ABSTRACT (Continued)

for the existence of a steady-state disturbance-utilizing control law are derived. It is shown that the steady-state control gains, when they exist, are solutions of certain matric algebraic equations. Steady-state gain expressions are derived for several first- and second-order plants. Results are presented for the cases of optimal disturbance-utilizing control of an air-defense interceptor missile and of a missile homing on a fixed target, which demonstrate performance superior to that obtained by a conventional "optimal" linear-quadratic missile controller.

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LIST OF SYMBOLS

a Plant system dynamics scalar coefficient.

A Plant system dynamics coefficient matrix; dimension n by n.

$\bar{A}$ Augmented system matrix defined by

$$\bar{A} = \left[\begin{array}{c|c|c} A & O & FH \\ \hline O & E & O \\ \hline O & O & D \end{array}\right]$$

A_1 A partition block of $\bar{A}$, defined by

$$A_1 = A$$

A_2 A partition block of $\bar{A}$, defined by

$$A_2 = \left[\begin{array}{c|c} E & O \\ \hline O & D \end{array}\right]$$

A_3 A partition block of $\bar{A}$, defined by

$$A_3 = \left[\begin{array}{c|c} O & FH \end{array}\right]$$

A_{CL} A "closed-loop" matrix defined by

$$A_{CL} = (A - BR^{-1}B^T K_X)$$

A_{xc} The matrix defined by the Kronecker sum

$$A_{xc} = A_{CL} X I_{\nu} + I_n X E^T$$

A_{xz} The matrix defined by the Kronecker sum

$$A_{xz} = A_{CL} X I_{\rho} + I_n X D^T$$

.✓ Assistance of disturbances in disturbance - utilizing control problems.

b Plant input scalar coefficient.

B Plant input coefficient matrix; dimension n by r.

LIST OF SYMBOLS (Continued)

$\overline{B}$ Augmented input matrix defined by

$$\overline{B} = \begin{bmatrix} B \\ \overline{O} \\ \overline{O} \end{bmatrix}$$

B_1 A partition block of $\overline{B}$, defined by

$$B_1 = B$$

$\mathscr{B}$ Burden of disturbances in disturbance - utilizing control problems.

c (1) State vector of a set-point or servo-tracking dynamic process; dimension ν by 1.

(2) Plant output scalar coefficient

(3) Piecewise-constant coefficient in a general expression for a disturbance waveform

C Plant output coefficient matrix; dimension n by p.

$\overline{C}$ Augmented output matrix defined by

$$C = \left[-C \mid G \mid 0 \right]$$

C_{xc} The ($n\nu$ by 1) vector whose elements are the elements of the matrix $C^T QG$

C_{xz} The ($n\rho$ by 1) vector whose elements are the elements of the matrix $K_x FH$.

Disturbance system dynamics coefficient matrix; dimension ρ by ρ .

$\mathscr{D}$ Composite state reconstructor design parameter defined by

$$\mathscr{D} = \left[T_{12} \mid T_{22} \right] \left(\left[\frac{A}{O} \middle| \frac{FH}{D} \right] \left[\frac{T_{12}}{T_{22}} - \frac{\dot{T}_{12}}{\dot{T}_{22}} \right] \right)$$

e (1) Set-point error $x_{sp}-x$ or $y_{sp}-y$.

(2) Servo-command error x_c-x or y_c-y.

LIST OF SYMBOLS (Continued)

E Set-point or servo-command dynamic system matrix; dimension (ν by ν).

EAU Control fuel consumption measure. In scalar control problems, defined by

$$EAU = \frac{1}{2} \int_{t_o}^{T} |u(t)| dt .$$

In vector control problems, defined by

$$EAU = \frac{1}{2} \int_{t_o}^{T} \left(|u_1(t)| + |u_2(t)| \right) dt .$$

EU Control energy consumption measure defined by

$$EU = \frac{1}{2} \int_{t_o}^{T} u^T(t) \; u(t) \; dt$$

$\mathscr{E}_A$ Angle error effectiveness defined by

$$\mathscr{E}_A = \frac{(\text{angle error at } t=T)_{LQ} - (\text{angle error at } t=T)_{DUC}}{(\text{angle error at } t=T)_{LQ}} \times 100\%$$

$\mathscr{E}_E$ Control energy effectiveness defined by

$$\mathscr{E}_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

$\mathscr{E}_M$ Miss distance effectiveness in scalar-system regulator problems, defined by

$$\mathscr{E}_M \triangleq \frac{|x_{sp} - x(T)|_{LQ} - |x_{sp} - x(T)|_{DUC}}{|x_{sp} - x(T)|_{LQ}} \times 100\%$$

LIST OF SYMBOLS (Continued)

$\mathscr{E}_{M_1}$ Miss distance effectiveness along x_1 in second-order regulator problems, defined by

$$\mathscr{E}_{M_1} \triangleq \frac{|x_{1_{SP}} - x_1(T)|_{LQ} - |x_{1_{SP}} - x_1(T)|_{DUC}}{|x_{1_{SP}} - x_1(T)|_{LQ}} \times 100\%$$

$\mathscr{E}_{M_2}$ Miss distance effectiveness along x_2 in second-order regulator problems, defined by

$$\mathscr{E}_{M_2} \triangleq \frac{|x_{2_{SP}} - x_2(T)|_{LQ} - |x_{2_{SP}} - x_2(T)|_{DUC}}{|x_{2_{SP}} - x_2(T)|} \times 100\%$$

$\mathscr{E}_{M_h}$ Miss distance effectiveness (horizontal) defined by

$$\mathscr{E}_{M_h} = \frac{|x_1(T)_{LQ}| - |x_1(T)_{DUC}|}{|x_1(T)_{LQ}|} \times 100\%$$

$\mathscr{E}_{M_v}$ Miss distance effectiveness (vertical) defined by

$$\mathscr{E}_{M_v} = \frac{|x_3(T)_{LQ}| - |x_3(T)_{DUC}|}{|x_3(T)_{LQ}|} \times 100\%$$

$\mathscr{E}_{MD}$ Miss distance effectiveness. In "small-LOS-angle" problems, defined by

$$\mathscr{E}_{MD} = \frac{|x_1(T)|_{LQ} - |x_1(T)|_{DUC}}{|x_1(T)|_{LQ}} \times 100\%$$

LIST OF SYMBOLS (Continued)

In general planar problems, defined by

$$\mathcal{E}_{MD} = \frac{(MD)_{LQ} - (MD)_{DUC}}{(MD)_{LQ}} \times 100\%$$

$\mathcal{E}_T$ Total effectiveness defined by

$$\mathcal{E}_T = \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

F Disturbance input matrix; dimension n by p.

G Output matrix in dynamic model of set-point and servo-tracking processes; dimension m by ν.

H Output matrix in dynamic model of disturbances; dimension p by ρ.

$\mathcal{K}$ Composite state reconstructor design parameter defined by

$$\mathcal{K} = \left[C \mid 0\right]\left(\left[\begin{array}{c|c} \dfrac{A}{0} & \dfrac{FH}{D} \end{array}\right]\left[\dfrac{T_{12}}{T_{22}}\right] - \left[\dfrac{\dot{T}_{12}}{\dot{T}_{22}}\right]\right)$$

J Quadratic performance index defined by

$$J = \tfrac{1}{2}e^T(T)Se(T) + \tfrac{1}{2}\int_{t_o}^{T}\left[e^T(t)Q(t)e(t) + u^T(t)R(t)u(t)\right] dt$$

k_x Scalar gain (see definition of K_x)

k_{xc} Scalar gain (see definition of K_{xc})

k_{xz} Scalar gain (see definition of K_{xz})

LIST OF SYMBOLS (Continued)

k_c Scalar gain (see definition of K_c)

k_{cz} Scalar gain (see definition of K_{cz})

k_z Scalar gain (see definition of K_z)

K_x (n by n) matrix solution of the first of the set of six unilaterally-coupled differential equations

K_{xc} (n by ν) matrix solution of the second of the set of six unilaterally-coupled differential equations

K_{xz} (n by ρ) matrix solution of the third of the six unilaterally-coupled differential equations

K_c (ν by ν) matrix solution of the fourth of the set of six unilaterally-coupled differential equations

K_{cz} (ν by ρ) matrix solution of the fifth of the set of six unilaterally-coupled differential equations

K_z (ρ by ρ) matrix solution of the last of the set of six unilaterally-coupled differential equations

L State coupling matrix in general output equation for disturbance w; dimension (p by n)

m (1) Dimension of output vector y in plant output equation.

(2) Mass of missile (slugs)

M State coupling matrix in disturbance state differential equation; dimension (ρ by n)

MD Miss distance (feet) at terminal time in zero set-point regulator problems defined by

$$MD = \sqrt{x_1^2(T) + x_3^2(T)}$$

LIST OF SYMBOLS (Continued)

n Dimension of plant state vector x.

p Dimension of disturbance vector w.

$\bar{P}$ The partioned matrix defined by

$$\bar{P} = \begin{bmatrix} K_x(t) & K_{xc}(t) & K_{xz}(t) \\ K_{xc}^T(t) & K_c(t) & K_{cz}(t) \\ K_{xz}^T(t) & K_{cz}^T(t) & K_z(t) \end{bmatrix}$$

q Scalar state-weighting parameter in integral term of quadratic performance index J.

Q State-weighting matrix in integral term of quadratic performance index J; dimension (n by n).

$\bar{Q}$ State-weighting matrix in integral term of the form of the quadratic performance index J used with the composite state vector x; dimensions ($n + \nu + \rho$ by $n + \nu + \rho$). Definition: $\bar{Q} = C^T QC$.

r (1) Dimension of control vector u.

(2) Scalar control weighting parameter in integral term of quadratic performance index J.

R Control weighting matrix in integral term of quadratic performance index J; dimension (r by r).

s Scalar weighting parameter on state at terminal time in quadratic performance index J.

S Weighting matrix on state at terminal time in quadratic performance index J; dimension (n by n).

$\bar{S}$ Weighting matrix on state at terminal time in the form of the quadratic performance index J used with the composite state vector $\tilde{x}$; dimension ($n + \nu + \rho$ by $n + \nu + \rho$). Definition: $\bar{S} = \bar{C}^T S\bar{C}$.

t Time (seconds)

LIST OF SYMBOLS (Continued)

t_0 Initial time (seconds) in regulator and servo-tracking problems.

T (1) Specified terminal time in regulator and servo-tracking problems (seconds).

(2) Symbol for transpose of a matrix or vector

T_{12} and T_{22} Matrices used in state reconstructor design with dimensions $n \times (n + \rho - m)$ and $\rho \times (n + \rho - m)$, respectively, chosen to satisfy

$$\left[C \,\middle|\, 0 \right] \left[\frac{T_{12}}{T_{22}} \right] = 0 \; ; \; \text{rank} \left[\frac{T_{12}}{T_{22}} \right] = n + \rho - m$$

$\bar{T}_{12}$ and $\bar{T}_{22}$ Matrices used in state reconstructor design, defined by

$$\bar{T}_{12} = (T_{12}{}^{T} T_{12} + T_{22}{}^{T} T_{22})^{-1} T_{12}{}^{T}$$

$$\bar{T}_{22} = (T_{12}{}^{T} T_{12} + T_{22}{}^{T} T_{22})^{-1} T_{22}{}^{T}$$

U Control variable in dynamic plant equations.

U_1 Horizontal control acceleration (ft/sec^2).

U_2 Vertical control accelerator (ft/sec^2).

U^o The optimal control in regulator and servo-tracking problems.

U_L Optimal control force component along longitudinal axis of missile (pounds).

U_N Optimal control force component along axis normal to longitudinal axis of missile (pounds).

U_{RES} Optimal control force resultant defined by

$$U_{RES} \triangleq \sqrt{U_L^2 + U_N^2}$$

LIST OF SYMBOLS (Continued)

$\mathcal{U}$ Utility of disturbance, defined by

$$\mathcal{U} \triangleq V\Big|_{w(t)\,\equiv\,0} - V\Big|_{w(t)\,\not\equiv\,0}$$

or by

$$\mathcal{U} = \mathcal{A} - \mathcal{B}$$

V The solution of the Hamilton-Jacobi-Bellman equation; the value J^o of the performance index J obtained under optimal control $u = u^o$.

w Disturbance input vector; dimension p by 1.

x Plant state vector; dimension n by 1.

$\hat{x}$ Estimate of plant state vector x.

$\tilde{x}$ Augmented state vector, defined by

$$\tilde{x} = \begin{pmatrix} x \\ \hline z \end{pmatrix}$$

y Plant output vector; dimension m by 1.

z State of disturbance dynamic process; dimension ρ by 1.

$\hat{z}$ Estimate of disturbance state z.

α Weighting for exponentially-decaying disturbance.

α_h Angle of reference line-of-sight relative to the ground.

β Coefficient in scalar differential equation representing a general dynamic model for a disturbance.

Γ Auxiliary matrix in disturbance-absorbing problems, satisfying

$$F(t)\left[H(t)\,|\,L(t)\right] \equiv B(t)\Gamma(t)$$

ε Error variable in state reconstructor problems.

LIST OF SYMBOLS (Continued)

ε_x Error in estimate of plant state vector x.

ε_z Error in estimate of distubance state z.

θ_{cz} Descriptor, in the c-z plane, of the size of the domain of positive utility.

θ_{xz} Descriptor, in the x-z plane, of the size of the domain of positive utility.

λ Eigenvalue symbol.

ν Dimension of state vector c.

μ Sparse sequence of impulses.

ξ Auxiliary variable in state reconstructor, satisfying the differential equation

$$\dot{\xi} = (\mathcal{A} + \Sigma \mathcal{K})\xi + \Psi y + \Omega u$$

ρ Dimension of disturbance vector z.

σ Sparse sequence of impulses.

Σ Intermediate design parameter in state-reconstructor example

τ Backward time (seconds) defined by

$$\tau = T - t$$

Ψ Intermediate design parameter, in state reconstructor example, defined by

$$\Psi = (\bar{T}_{12} + \Sigma C)(A C^{\#T} - \dot{C}^{\#T}) - (\mathcal{A} + \Sigma \mathcal{K}) \Sigma + \dot{\Sigma}$$

ω A sparse impulsive sequence consisting of impulses and doublets.

Ω An intermediate design parameter, in state reconstructor example, defined by

$$\Omega = (\bar{T}_{12} + \Sigma C)B$$

$(\cdot)$ Time-derivative of parameter in parentheses.

LIST OF SYMBOLS (Concluded)

$(\bar{\ })$ (1) Steady-state value of parameter in parentheses when the parameter is

$$k_x,\ k_{xc},\ k_{xz},\ k_c,\ k_{cz},\ k_x,\ K_x,\ K_{xc},\ K_{xz},\ K_c,\ K_{cz},\ K_z,\ \dot{k}_c,$$
$$\dot{k}_{cz},\ \dot{k}_z,\ \theta_{xz} \text{ or } \theta_{cz}\ .$$

(2) Indication that the parameter in parentheses is a partitioned matrix when the parameter is C, S, Q, A, B or P.

$(\hat{\ })$ Estimate of the parameter in parentheses.

$\nabla(\ )$ Gradient of the parameter in parentheses.

$(\ )^{\#}$ Generalized Moore-Penrose inverse of parameter in parantheses.

CHAPTER I

INTRODUCTION: THE GENERAL THEORY OF DISTURBANCE-ACCOMMODATING CONTROL

1.1 Summary of Chapter I

This chapter discusses general aspects of the theory of disturbance-accommodating control. The nature of disturbances, the distinction between noise and disturbances, and categories of disturbances are considered. A general discussion of optimal control theory for the control problem with disturbances is followed by presentation of an approach to optimal control in the case where the disturbances have "waveform structure." Finally, the theory of optimal control for the linear-quadratic regulator with disturbances is introduced, and the three primary modes of disturbance accommodation are discussed: the cancellation mode, the minimization mode, and the maximum utilization mode.

1.2 Disturbances in Control Problems; Their Nature and Philosophies of Accommodation

Controlled systems are typically subjected to uncontrolled inputs arising from a variety of sources. These uncontrolled inputs, referred to as disturbances, usually occur at unpredictable times and are commonly viewed as undesirable. They may be broadly classified as either noise-type disturbances or disturbances with waveform structure [1].

Uncontrolled inputs, which have completely erratic, random characteristics (i.e., no significant degree of regularity), are classified as noise-type disturbances. Thermal noise encountered in radar and radio receivers is an example of a noise-type disturbance. Disturbances of this type are often modeled by their statistical moments and spectral density properties. The fields of stochastic optimal estimation and control are concerned almost entirely with noise-type disturbances, and several texts [2]-[4] provide excellent coverage of these topics.

On the other hand, many uncontrolled inputs have "waveform structure" - their waveshape is describable as a weighted combination of certain known basis functions. For example, they may consist of weighted linear combinations of steps, ramps, exponentials or other functions, even though the specific values of the weighting coefficients or the times at which they change value may be unknown. Such inputs will be classified as waveform-type disturbances. For example, wind gusts acting on a missile may be classified as a waveform-type disturbance.

A further classification of disturbances may be made by recognizing waveform-type disturbances as being either natural or command disturbances. Some examples of natural disturbances are wind forces on aircraft, fluctuating loads on power generators and drift in an amplifier. An example of a command disturbance arises in connection with a set-point regulator problem, in which the primary control task is to regulate the state x(t) to a given set-point x*. Consider the usual linear state-variable model of a controlled system:

$$\dot{x} = A\,x + B\,u \tag{1.1}$$

where x is the state vector, A is the "plant" matrix, B is the input matrix, and u is the control vector applied to the system. The "set-point error" is defined as $x_e = x^* - x$ and, using Equation (1.1), the dynamics of $x_e(t)$ are found to be governed by

$$\dot{x}_e = A\,x_e - B\,u - A\,x^* \tag{1.2}$$

Therefore, the control objective in terms of Equation (1.2) is to regulate the error state x_e to zero. The term Ax^* is an "uncontrolled" input and thus has the effect of a known external disturbance in the model Equation (1.2). It is therefore evident that a controlled system represented by the conventional model Equation (1.1) fails to account for the presence of such command disturbances. A similar disturbance arises in the servo-tracking problem associated with Equation (1.1) wherein a prescribed servo-command function results in known, time-varying external disturbances. Thus, even in the absence of "natural" disturbances, there is a need to account for command disturbances in control system models.

Traditionally, the uncontrolled inputs associated with control system design have been viewed as being detrimental to the task of the control system. For example, in classical control system design, the frequency response of the overall closed-loop system is often shaped to attempt to filter-out noise and disturbances, while maintaining desired stability and accuracy performance. Classical control design approaches have resulted in such design schemes as "integral control", "feedforward control," and the notch filter to minimize the effects of noise and disturbances. On the other hand, there are practical situations in which the effects of disturbances are not always detrimental to achieving control objectives. For example, in a missile intercept problem, where the primary control objective is to drive the missile so that the position of the missile coincides with that of the target, wind gusts that force the interceptor missile to move in the direction of the target may be constructively used to aid in the control task. In particular, the presence of the wind disturbance may actually reduce the interceptor control energy and the time required to intercept the target. The concept of harnessing "free" energy from winds, tides, etc. has, of course, been used in applications other than control systems, and will no doubt see extensive further development.

The application of modern control theory techniques permits the consideration of three modes of disturbance accommodation:

(a) exact cancellation of the effect of the disturbance on the control system,

(b) the "best" approximation to cancellation of the effect of the disturbance (when exact cancellation is not achievable), and

(c) optimal utilization of the disturbance in accomplishing the control objectives.

In addition, combinations of these three modes may be used in particular applications.

The theory to be developed in the present study assumes that the disturbances might not be directly measurable. In fact, in the typical case, only the commands and the plant output y(t) are available as measurements to the controller, where y(t) is a known algebraic function of time and the states of the plant.

1.3 Optimal Control of Dynamical Systems in the Presence of Disturbances; A General Approach

A fundamental difficulty arises when an optimal control problem is formulated to include uncontrolled inputs such as disturbances. Johnson [5] showed that the standard approach via the Pontryagin maximum principle is effective only if the time-behavior of the disturbance function is entirely known *a priori*. Unfortunately, this is not a situation enjoyed in practice, since the behavior of disturbances is almost never precisely known ahead of time.

One alternative to this standard optimal control approach is the stochastic-control method, which treats disturbances as noise and utilizes statistical moments (mean, covariance, etc.) to characterize the disturbance time-behavior. The underlying assumption of the stochastic approach is that all disturbance functions with the same mean, covariance, etc. are modelled alike, ignoring any additional information, such as waveform structure, that may be available. Statistical moments, such as the mean and covariance, are based on averages over relatively long time intervals. High-performance control system designs, however, often require short-term disturbance behavior patterns for effective operation. For example, the long-term average value wind-gust forces on an aircraft may be very close to zero, but effective control of the aircraft in the presence of wind gusts requires short-term behavior information about the disturbance. The characterization of disturbances solely by statistical properties is justifiable in control system design only when no waveform-mode characterization is possible; that is, when the disturbance is essentially noise.

1.4 Optimal Control in the Presence of Disturbances Having Waveform Structure

1.4.1 Disturbance Modeling. Johnson [1] introduced the concept of modeling uncertain waveform-type disturbances by giving a differential equation that the disturbance is known to satisfy. The uncertain disturbance is described, in this approach, as a linear combination of functions:

$$w(t) = c_1 f_1(t) + c_2 f_2(t) + \ldots + c_m f_m(t) \qquad (1.3)$$

where the coefficients c_i are piecewise constant, but unknown, and the functions $f_i(t)$, called "basis functions," are known functions of time which characterize the possible modes of the disturbance.

Suppose that the differential equation

$$\frac{d^\rho w}{dt^\rho} + \beta_\rho \frac{d^{\rho-1} w}{dt^{\rho-1}} + \ldots + \beta_2 \frac{dw}{dt} + \beta_1 w = \omega(t) \qquad (1.4)$$

(where the coefficients β_j are constants and $\omega(t)$ is an impulsive function consisting of delta functions, doublets, etc.) has Equation (1.3) as its solution. Then the effect of $\omega(t)$ will be to cause the coefficients c_i to jump in value in a piecewise constant fashion at the completely unknown arrival times of $\omega(t)$.

As an example, the piecewise constant disturbance

$$w_1(t) = c \qquad (1.5)$$

where c is unknown and changes its value at unknown times in a piecewise-constant fashion, clearly satisfies the differential equation

$$\frac{dw_1}{dt} = \sigma(t) \qquad (1.6)$$

where $\sigma(t)$ denotes a sparse sequence of randomly arriving impulses which cause the piecewise constant amplitude of the disturbance $w_1(t)$ to change to a new value every once in a while.

Similarly, a disturbance consisting of a linear combination of constant segments and linear ramps:

$$w_2(t) = c_1 + c_2 t \tag{1.7}$$

clearly satisfies the equation

$$\frac{d^2 w_2}{dt^2} = \omega(t) \tag{1.8}$$

where the impulsive sequence $\omega(t)$ consists of isolated impulses and doublets which cause the c_1 and c_2 to change at unknown, random times.

In the general case, the basis functions $f_1(t)$, $f_2(t)$, . . ., $f_m(t)$ in Equation (1.3) may be constants, ramps, polynomials, exponentials, sinusoidal terms, etc. (and linear combinations of these), corresponding to the mode content of the particular disturbance of interest. The modeling approach will then be to find a differential equation of the general form as Equation (1.4) which has the disturbance $w(t)$ as its solution.

This approach will be used to represent realistic disturbances in the present study. It will often be useful to view the disturbance as the "output" of a generally nonlinear dynamic process

$$\dot{z} = \mathcal{D}(z, x, t) + \sigma(t) \tag{1.9}$$

$$w = \mathcal{W}(z, x, t), \; t_0 \leq t \leq T \tag{1.10}$$

where z is the disturbance "state" vector, x is the plant state vector, w is the disturbance vector and σ is a vector whose elements are sequences of impulse functions. The functions $\mathcal{D}$ and $\mathcal{W}$ are, in general, time-varying, non-linear and may involve the plant state x (representative of a plant-dependent disturbance process). Since the possible modes of the disturbance are assumed to be known *a priori*, the functions $\mathcal{D}$ and $\mathcal{W}$ are known, but the vector impulse sequence $\sigma(t)$ is completely unknown.

In the case of the disturbance $w_2(t)$, Equation (1.7), (a linear combination of constant levels and ramps which satisfies Equation (1.8)), the disturbance process is the linear system

$$\dot{z} = D\,z + \sigma(t) \tag{1.11}$$

$$w = H\,z \tag{1.12}$$

where z and $\sigma(t)$ are 2-vectors, w is a scalar and D and H are defined by

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{1.13}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{1.14}$$

It should be noted that the dynamic process, Equations (1.9) and (1.10), seen as generating the disturbance w(t), is a fictitious process. Nevertheless, the actual processes which generate typical disturbances such as wind gusts, load variations, drifts, and biases can be accurately represented by this type of model.

1.4.2 Optimal Control. The state model, Equation (1.9), of the disturbance process can be combined with the typical state model of the plant dynamics, resulting in the general expressions

$$\dot{x} = \mathscr{F}(x,\ t,\ u(t)\ ,\ \mathscr{W}(z,\ x,\ t)) \tag{1.15}$$

$$\dot{z} = \mathscr{D}(z,\ x,\ t) + \sigma(t) \tag{1.16}$$

Johnson has shown [5] that the optimal control u^o, which minimizes

$$J\left[u;\ x_o,\ t_o,\ T\right] = G(x(T),\ T) + \int_{t_o}^{T} (L(x(t),t,u(t))\ dt \tag{1.17}$$

subject to the combined system Equations (1.15) and (1.16), and assuming $\sigma(t) \equiv o$, can be expressed as

$$u^o = u^o(x,\ z,\ t)\ . \tag{1.18}$$

That is, the optimal control at time t is a function of the current state x(t) of the plant and the current state z(t) of the disturbance. This result may be contrasted with that obtained by the conventional optimization approach, which gives the optimal control as a function of the plant state $x(t)$ alone. The control Equation (1.18), which accounts for the presence of disturbances, was derived under the assumption that the impulse sequence $\sigma(t)$ was identically zero. In fact, $\sigma(t)$ is sparsely populated and unknown *a priori*; and, therefore, its effect could be viewed as a sequence of unknown initial conditions $z(t_o)$ imposed on the model Equation (1.16). A corollary to this viewpoint (stated as a conjecture in [5]) is that the control $u^o(x,z,t)$ given by Equation (1.18) is "optimal" also for the case where the sparsely populated impulsive sequence $\sigma(t)$ is present.

Realization of the control law Equation (1.18) requires that real-time, current values of the states (x,z) be made available to the controller, through either direct measurements or use of an observer. A discussion of the implementation of plant/disturbance state observers may be found in [1], [5], [6], and in Appendix A of this dissertation.

1.5 Optimal Control of the Linear-Quadratic Regulator with Disturbances

1.5.1 The System Model.

A special case of the optimal control theory discussed in Subsection 1.4.2 is the linear-quadratic regulator with disturbances present. Johnson has shown in [1], [5], [6] and [7] how the disturbance accommodating theory applies to the set-point regulator and servo-tracking control problems in which the plant dynamics are modeled as:

$$\dot{x} = A(t)x + B(t)u(t) + F(t)w(t) \tag{1.19}$$

$$y = C(t)x \tag{1.20}$$

where x, u and w are vectors of dimension n, r and p, respectively, and $n \geq r \geq p$. The disturbance process (1.9), (1.10) is modeled by the linear system:

$$w(t) = H(t)z + L(t)x \tag{1.21}$$

$$\dot{z} = D(t)z + M(t)x + \sigma(t) \tag{1.22}$$

where z is a ρ-dimensional vector.

1.5.2 The Cancellation Mode of Accommodation. The problem of regulating the state x to a set-point, while attempting to completely cancel the disturbances may be considered by splitting the control into two parts [6]:

$$u = u_C + u_R \tag{1.23}$$

where u_C is the control required to perform disturbance cancellation and u_R is the control required to drive x to the desired set-point. For the special case of zero state set-point, the control objective is to minimize the quadratic functional

$$J(u) = \int_{t_o}^{T} \left[x^T(t)Q(t)x(t) + u_R^T(t)R(t)\, u_R(t)\right] dt \tag{1.24}$$

subject to the terminal condition $x(T) = 0$ and to the plant dynamics Equations (1.19) and (1.20), and in the face of any possible disturbance $w(t)$ produced by Equations (1.21) and (1.22), where $Q(t)$ and $R(t)$ are known positive-definite, symmetric matrices on the interval $[t_o, T]$. In (1.24) T is the terminal time, and may be fixed a priori or may be unspecified.

The disturbance-accommodating control, if it exists, must be such that the term $F(t)w(t)$ in Equation (1.19) is exactly cancelled by control action $B(t)\,u_c(t)$. That is, the required control component u_c is of the form

$u_c = \phi_c\,(x,t,w)$

where

$$B(t)\phi_c(x,\ t,w) + F(t)w(t) \equiv 0 \qquad (1.25)$$

for all realizable values of $w(t) = Hz + Lx$. Equation (1.25) can be satisfied if, and only if, the column range space of $F(t)[H(t)|L(t)]$ lies within the column range space of $B(t)$. That is,

$$F(t)\left[H(t)\,|\;L(t)\right] \equiv B(t)\Gamma(t) \qquad (1.26)$$

for some matrix $\Gamma(t)$, or, equivalently,

$$\text{Rank}\left[B(t)\,|F(t)\left[H(t)\,|L(t)\right]\right] \equiv \text{Rank}\left[B(t)\right],\ t_o \leq t \leq T \qquad (1.27)$$

If Equation (1.26) is satisfied, then Equation (1.25) can be satisfied by choosing

$$u_c = \phi_c(x,\ t,\ w) = -\ \Gamma(t)\left(\frac{z}{x}\right) \tag{1.28}$$

such that complete cancellation of the disturbance is obtained. Substituting Equations (1.23) and (1.28) into Equation (1.19) then gives

$$\dot{x} = A(t)x + B(t)u_R(t) \tag{1.29}$$

Conventional linear-quadratic regulator theory for the reduced problem Equation (1.29) (for example, see Athans and Falb [8]) gives the optimal control u_R for Equations (1.24) and (1.29) in the familiar state-feedback form

$$u_R(t) = K(t)x(t) \tag{1.30}$$

where the feedback gain matrix K(t) satisfies a particular matrix Riccati differential equation. The complete control u^o is the superposition of Equations (1.28) and (1.30):

$$u^o(x,\ z,\ t) = \left[K(t) - \Gamma_2(t)\right]\ x - \Gamma_1(t)\ z \tag{1.31}$$

where $\Gamma = \left[\Gamma_1 | \Gamma_2\right]$.

Implementation of this optimal control law employs an estimator to generate estimates $\hat{x}$ and $\hat{z}$ of the states x and z from measurements of the output y. Johnson has shown [5] that these estimates may be obtained from the composite estimator described by

$$\begin{pmatrix} \dot{\hat{x}} \\ \dot{\hat{z}} \end{pmatrix} = \left[\begin{array}{c|c} A(t) + B(t)K(t) & 0 \\ \hline M(t) & D(t) \end{array}\right] \begin{pmatrix} \hat{x} \\ \hat{z} \end{pmatrix} + \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right] \left[C\hat{x} - y\right] \tag{1.32}$$

where the matrices $K_1(t)$ and $K_2(t)$ are chosen to make the estimation error

$$\varepsilon = \begin{pmatrix} x(t) \\ \hline z(t) \end{pmatrix} - \begin{pmatrix} \hat{x}(t) \\ \hline \hat{z}(t) \end{pmatrix} \tag{1.33}$$

settle toward zero quickly between arrival times of the isolated impulses of $\sigma(t)$. The term y in Equation (1.32) represents the measurement of the output of the actual plant, and $\hat{x}$ and $\hat{z}$ are the resulting on-line, real-time estimates which are then used in the implementable control:

$$\hat{u}_o\ (\hat{x},\ \hat{z},\ t) = \left[K(t)\ \ -\Gamma_2(t)\right]\ \ \hat{x} - \Gamma_1(t)\,\hat{z} \quad . \tag{1.34}$$

The resulting controller is called a "disturbance-absorbing controller" and has interesting features which may be compared with results from classical design approaches. For the case of a piecewise constant disturbance, a proportional-plus-integral controller is obtained from Equation (1.34); for disturbances that are represented by higher-order polynominals, multiple-integral feedback structures are obtained. In the case of sinusoidal disturbances, with completely unknown phase and amplitude, expression Equation (1.34) produces the classical notch-filter effect for the closed-loop system.

1.5.3 <u>The Minimization Mode of Accommodation</u>. Complete cancellation of the effects of the disturbance on the plant dynamics may not be possible - it may be mathematically impossible to find a $\Gamma(t)$ satisfying Equation (1.26). If this is the case, then $u_c(t)$ may be chosen to <u>minimize</u> the effects of the disturbance on the plant behavior, in some specified sense. One approach is to minimize the norm

$$||B(t)u_c + F(t)w(t)|| \quad . \tag{1.35}$$

The vector u_c which minimizes Equation (1.35) is not unique, in general; but, if one chooses the $u^o{}_c$ which <u>itself</u> has minimum norm, then that $u^o{}_c$ is unique, and is given by

$$u_c^o = -B^{\#}(t)F(t)w(t) \tag{1.36}$$

where $B^{\#}(t)$ is the Moore-Penrose generalized inverse of B(t) [6], [9]. If the rank of B is equal to r (the dimension of the control $u^o{}_c$), then $B^{\#}$ has the specific form

$$B^{\#}(t) = \left[B^T(t)B(t)\right]^{-1} B^T(t) \tag{1.37}$$

The implementation of the control Equation (1.36) requires on-line, real-time estimates of x and z; and the modified composite state observer equations are described in [6]. The computation of $u_R(t)$ is performed as for the case of complete cancellation.

1.5.4 The Maximum Utilization Mode. Disturbances are not necessarily detrimental to the achieving of control system objectives. Although numerous approaches have been developed for cancelling or minimizing disturbances, the idea of utilizing disturbances in control systems is a relatively recent development [1], [5], [6], [7]. Constructive utilization of disturbances can lead to reduced control energy and reduced time required to bring the plant state to a required set-point objective. Likewise, in servo-tracking problems, disturbances may be constructively utilized to assist the control in guiding the plant output y(t) to faithfully "follow" a time-varying command function $y_{c(t)}$.

Maximum utilization of a disturbance w(t) having waveform structure can be achieved by employing optimal control theory to design the controller. Although this is virtually impossible using classical control system design approaches, it is relatively straightforward with modern optimal control theory. The key to obtaining maximum utilization of disturbances is to choose a performance index J so that, when J is minimized with respect to the control u(t), the primary control objective is accomplished and maximum use of the disturbance w(t) is achieved. For example, if the primary control objective is to regulate the plant state $x(t)$ to zero, a secondary objective may be to use as little control energy as possible. One may be able to achieve these objectives by choosing a quadratic-type performance index as

$$J = \frac{1}{2} x^T(T)\, Sx(T) + \frac{1}{2} \int_{t_o}^{T} \left[x^T(t)\, Qx(t) + u^T(t) Ru(t)\right] dt \tag{1.38}$$

where S and Q are given symmetric non-negative definite matrices. S + Q is positive definite, R is a positive-definite matrix, and the terminal time T is specified. Note that, in this design, the control $u(t)$ is <u>not</u> split into components as was the case in Equation (1.23). The presence of the positive definite matrix R encourages the effective utilization of any "free" energy available in the disturbance. This approach was used for a special application in linear systems in the work of Johnson and Skelton [13], and was subsequently generalized in the work of Johnson [6].

In the next chapter it will be seen that the disturbance utilizing problem can be formulated as a linear-quadratic regulator problem by using the augmented vector

$$\tilde{x} = \left(\frac{x}{z}\right) \tag{1.39}$$

which is a composite of the state vectors of the plant and the disturbance process. The composite system equation may be written by using $\tilde{x}$ and the plant and disturbance dynamic

Equations (1.19), (1.21) and (1.22), with $L(t) = 0$, as follows:

$$\dot{\tilde{x}} = \begin{pmatrix} \dot{x} \\ \dot{z} \end{pmatrix} = \left[\begin{array}{c|c} A & FH \\ \hline 0 & D \end{array}\right] \tilde{x} + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] u + \begin{pmatrix} 0 \\ \sigma \end{pmatrix} \quad . \tag{1.40}$$

The performance index Equation (1.38) can be written in the equivalent form

$$J = \tfrac{1}{2}\tilde{x}(T)\bar{S}\tilde{x}(T) + \tfrac{1}{2} \int_{t_o}^{T} \left[\tilde{x}^T(t)\bar{Q}\tilde{x}(t) + u^T(t)Ru(t)\right] dt \tag{1.41}$$

where $\bar{S} = \bar{C}^T S\bar{C}$, $\bar{C} = [-C|0]$ and $\bar{Q} = \bar{C}^T Q\bar{C}$. It will be seen in Chapter II that the sparse sequence of impulses $\sigma(t)$ can be disregarded and the control which minimizes Equation (1.41) subject to Equation (1.40) can be found using standard linear-quadratic methods, resulting in the control

$$u^o = -R^1 B^T \left[K_x x + K_{xz} z\right] \tag{1.42}$$

which is a function of the states of the plant <u>and</u> of the disturbance process. It will also be seen that the time varying gain matrix $K_x(t)$ is the familiar gain term obtained as the solution of a certain matrix Riccati equation, as in the standard linear-quadratic regulator problem. The time-varying gain matrix K_{xz} will be found as the solution of a certain <u>linear</u> matrix equation which depends upon K_x and the parameters of the plant and disturbance processes. The derivation and properties of K_x, K_{xz} and additional matrices associated with the more general non-zero set-point and servo-tracking problems will be further examined in the next chapter.

CHAPTER II

DISTURBANCE-UTILIZING CONTROL; GENERAL THEORY FOR LINEAR-QUADRATIC PROBLEMS

2.1 Summary of Chapter II

This chapter presents a general theory for disturbance-utilizing control in the case of linear-quadratic regulator and servo-tracking problems. A description of the linear-quadratic regulator and servo-tracking control problems is given, followed by a discussion of the control objectives, problem formulation, and general solution of these problems in the context of disturbance-utilizing control. The computational and dynamic properties of the disturbance-utilizing control law are discussed in terms of its general behavior, steady-state solutions and methods for determining steady-state values. The concepts of burden, assistance and utility are presented and the properties of the utility function and domains of positive utility are examined in detail.

2.2 Description of the Linear-Quadratic Regulator and Servo-Tracking Control Problems

The concept of achieving maximum utilization of a disturbance by employing optimal control theory in the controller design was introduced in the previous chapter. The specific cases of obtaining maximum utilization of a disturbance in linear-quadratic regulator and servo-tracking control problems will now be described. The approach will be demonstrated by considering a specific example using the linear system model

$$\dot{x} = A(t)x + B(t)u(t) + F(t)\ w(t) \tag{2.1}$$

$$y = C(t)x \tag{2.2}$$

where x, u and w are n-, r-, and p-vectors, respectively, and $n \geq r \geq p$. The disturbance process will be assumed to be a special case of the model of Equations (1.21) and (1.22) with $L(t) \equiv 0$ and $M(t) \equiv 0$:

$$w(t) = H(t)z \tag{2.3}$$

$$\dot{z} = D(t)z + \sigma(t) \tag{2.4}$$

2.3 The Objectives of Disturbance-Utilizing Control Strategy

The primary objective of control is assumed to be either:

(a) regulation to a given state set-point x_{sp} or a given output set-point y_{sp}, or

(b) servo-tracking of a given state or output servo-command function $x_c(t)$ or $y_c(t)$.

The secondary control objective is to achieve the primary objective while minimizing the control energy associated with u(t). Control energy is assumed to be measured by the time-integral of the quadratic form $u^T(t)R(t)u(t)$, when R(t) is a given symmetric, positive-definite matrix.

These control objectives may be achieved [7] by considering the quadratic performance index

$$J = \frac{1}{2}e^T(T)Se(T) + \frac{1}{2}\int_{t_o}^{T}\left[e^T(t)Q(t)e(t) + u^T(t)R(t)u(t)\right]dt \tag{2.5}$$

where S and Q(t) are given symmetric non-negative definite matrices, S + Q is positive definite, the terminal time T is given and, $e = y_c(t) - y(t)$ (or $y_{sp} - y(t)$). The case of <u>state</u> set-points or <u>state</u> servo-commands may be considered by setting C = I in Equation (2.2). The control objective is achieved by minimizing J with respect to the control u(t), subject to the plant and disturbance Equations (2.1)-(2.4).

The set of expected servo-commands $y_c(t)$ or set-points y_{sp} are modeled by the equations

$$y_c(t) = G(t)\, c \tag{2.6}$$

$$\dot{c} = E(t)c + \mu(t) \tag{2.7}$$

where y_c is an m-vector, c is the ν by 1 state vector of the set-point or servo-command process, $c(t_0)$ is arbitrary and $\mu(t)$ is a sparse sequence of impulses. If set-points are being considered, y_c should be set equal to y_{sp}, a piecewise constant function, and $E(t)$ will be identically zero. For the case of servo tracking or set-point regulation, the fundamental necessary condition for perfect set-point or servo tracking, $e(t) \equiv 0$, is that the "trackability condition" $G(t) = C(t)\theta(t)$ be satisfied for some $\theta(t)$ (see references [10], [11], [12], [37]).

2.4 Formulation of the Disturbance-Utilization Optimal Control Problem

In this controller design approach, the control $u(t)$ is not split as it was in Equation (1.23). The presence of the positive penalty term $u^T(t)R(t)u(t)$ in the integrand of the performance index Equation (2.5) encourages the maximum utilization of the "free energy" of the disturbance $w(t)$ while achieving the primary control objective of set-point regulation or servo-tracking. The disturbance utilization linear-quadratic optimal control problem is formulated (for example, see Reference [7]) by using the augmented vector

$$\tilde{x} = \begin{pmatrix} x \\ \hline c \\ \hline z \end{pmatrix} \qquad (2.8)$$

which is a composite of the state vectors of the plant, the set-point or servo-command process, and the disturbance process. The composite system equation may be written by using $\tilde{x}$ and Equations (2.1), (2.4), and (2.7) to obtain:

$$\dot{\tilde{x}} = \begin{pmatrix} \dot{x} \\ \hline \dot{c} \\ \hline \dot{z} \end{pmatrix} = \left[\begin{array}{c|c|c} A & O & FH \\ \hline O & E & O \\ \hline O & O & D \end{array}\right] \tilde{x} + \begin{bmatrix} B \\ \hline O \\ \hline O \end{bmatrix} u + \begin{pmatrix} O \\ \hline \mu \\ \hline \sigma \end{pmatrix} \qquad (2.9)$$

The performance index Equation (2.5) can be written in the equivalent form

$$J = \frac{1}{2}\tilde{x}^T(T)\bar{S}\tilde{x}(T) + \frac{1}{2}\int_{t_0}^{T}\left[\tilde{x}^T(t)\bar{Q}(t)\tilde{x}(t) + u^T(t)R(t)u(t)\right]dt \qquad (2.10)$$

where $\bar{S} = \bar{C}^T S\bar{C}$, $\bar{C} = [-C|G|0]$ and $\bar{Q} = \bar{C}^T Q\bar{C}$.

For reasons discussed in Reference [6], the sparse sequences of impulses $\sigma(t)$ and $\mu(t)$ in Equation (2.9) may be disregarded, and the solution of Equations (2.9) and (2.10) can be found using standard linear-quadratic optimal control methods. The optimal disturbance-utilizing control is the control $u^o(t)$ which minimizes Equation (2.10) subject to the dynamic Equation (2.9).

2.5 General Solution of the Disturbance-Utilization Optimal Control Problem

The minimization of the performance index J Equation (2.10) subject to the composite dynamic system Equation (2.9) can be accomplished by using the Hamilton-Jacobi theory [4], [8]. If we define the special function $V(\tilde{x},t)$ to be the value of the performance index J when the optimal control $u^o(t)$ is employed, i.e.

$$V(\tilde{x}, t) = J(u^o; \tilde{x}, t, T) \; ; \quad \tilde{x}(t_o) = \tilde{x} \quad t_o = t \qquad (2.11)$$

it can be shown that the function $V(\tilde{x},t)$ satisfies the Hamilton-Jacobi-Bellman partial differential equation

$$\frac{\partial V}{\partial t} + \nabla_{\tilde{x}} V^T \bar{A}\tilde{x} - \tfrac{1}{2}\nabla_{\tilde{x}} V^T \bar{B} R^{-1} \bar{B}^T \nabla_{\tilde{x}} V + \tfrac{1}{2}\tilde{x}^T \bar{Q}\tilde{x} = 0 \qquad (2.12)$$

subject to the boundary condition

$$V(\tilde{x}, T) = \frac{1}{2}\tilde{x}^T\bar{S}\tilde{x} \tag{2.13}$$

where $\bar{Q}$ and $\bar{S}$ have already been defined, and $\bar{A}$ and $\bar{B}$ are defined by

$$\bar{A} = \left[\begin{array}{c|c|c} A & O & FH \\ \hline O & E & O \\ \hline O & O & D \end{array}\right] \tag{2.14}$$

$$\bar{B} = \left[\begin{array}{c} B \\ \hline O \\ \hline O \end{array}\right] \tag{2.15}$$

and

$$\nabla_{\tilde{x}}^T V = \left(\frac{\partial V}{\partial \tilde{x}_1}, \ldots, \frac{\partial V}{\partial \tilde{x}_N}\right), \quad N = n + \nu + \rho \tag{2.16}$$

It may be shown [8] that, if $u(t)$ is not constrained, T is specified, S and $Q(t)$ are non-negative definite, $S + Q$ is positive definite and $R(t)$ is positive definite, then an optimal control exists and is given by

$$u^o(t) = -R^{-1}(t)\bar{B}^T(t)\nabla_{\tilde{x}} V \quad . \tag{2.17}$$

The sought solution of the Hamilton-Jacobi-Bellman Equation (2.12) can be expressed as the symmetric non-negative definite quadratic form

$$V(\tilde{x}, t) = \frac{1}{2}\tilde{x}^T\bar{P}(t)\tilde{x} \tag{2.18}$$

where $\bar{P}(t) = \bar{P}^T(t) \geq 0$ is chosen to satisfy Equation (2.12). For mathematical simplification, it is convenient to partition $\bar{P}$ as the 3 x 3 block matrix:

$$\bar{P} = \left[\begin{array}{c|c|c} K_x(t) & K_{xc}(t) & K_{xz}(t) \\ \hline K_{xc}^T(t) & K_c(t) & K_{cz}(t) \\ \hline K_{xz}^T(t) & K_{cz}^T(t) & K_z(t) \end{array}\right] \quad ; \ \bar{P}(T) = \bar{S} \tag{2.19}$$

where the dimensions of the component matrices are denoted as follows:

$$\begin{array}{l} \left[K_x\right]_{n \times n} ; \left[K_{xc}\right]_{n \times \nu} ; \left[K_{xz}\right]_{n \times \rho} ; \\ \left[K_{xc}^T\right]_{\nu \times n} ; \left[K_c\right]_{\nu \times \nu} ; \left[K_{cz}\right]_{\nu \times \rho} ; \\ \left[K_{xz}^T\right]_{\rho \times n} ; \left[K_{cz}^T\right]_{\rho \times \nu} ; \left[K_z\right]_{\rho \times \rho} . \end{array} \tag{2.20}$$

Now if Equations (2.18) and (2.19) are substituted in the Hamilton-Jacobi-Bellman Equation (2.12), the result is a set of six unilaterally coupled matric differential equations which determine the individual blocks K_{ij} of the partitioned matrix $\bar{P}$. Those equations, with their specific terminal conditions are as follows:

$$\dot{K}_x = (-A + BR^{-1}B^TK_x)^TK_x - K_xA - C^TQC \quad ;K_x(T) = C^TSC \quad (2.21)$$

$$\dot{K}_{xc} = (-A + BR^{-1}B^TK_x)^TK_{xc} - K_{xc}E + C^TQG \;;K_{xc}(T) = -C^TSG \quad (2.22)$$

$$\dot{K}_{xz} = (-A + BR^{-1}B^TK_x)^TK_{xz} - K_xFH - K_{xz}D \quad ;K_{xz}(T) = 0 \quad (2.23)$$

$$\dot{K}_c = -(K_cE + E^TK_c) + K_{xc}{}^TBR^{-1}B^TK_{xc} - G^TQG \;;K_c(T) = G^TSG \quad (2.24)$$

$$\dot{K}_{cz} = -(K_{cz}D + E^TK_{cz}) + K_{xc}{}^T(BR^{-1}B^TK_{xz} - FH);K_{cz}(T) = 0 \quad (2.25)$$

$$\dot{K}_z = -(K_zD + D^TK_z) + K_{xz}{}^TBR^{-1}B^TK_{xz} - \left[(FH)^TK_{xz} + K_{xz}{}^TFH\right] \quad (2.26)$$

$$;K_z(T) = 0$$

These equations are independent of the initial conditions on the plant, disturbances, and commands, and can be solved by integrating in backward time, starting at $t = T$ and "advancing" to $t = t_o$.

Finally, using the fact that $\nabla_x \tilde{V} = \bar{P}x$, and substituting this relation in Equation (2.17), the optimal disturbance-utilizing control is obtained as

$$u^o = -R^{-1}B^T \left[K_xx + K_{xc}c + K_{xz}z\right] . \qquad (2.27)$$

Note that, for the zero set-point regulator with no disturbances present, Equation (2.27) reduces to the form familiar from the solution of the "conventional" linear-quadratic zero set-point regulator problem

$$u^o = -R^{-1}B^T K_x x \tag{2.28}$$

in which K_x is the solution of Equation (2.21), a matric Riccati differential equation; (for example, see [8]):

As in the other modes of disturbance accommodation, a composite state reconstructor will be used to provide estimates of x, c, and z for implementation of the control law Equation (2.27).

2.6 Computational Features and Dynamic Properties of the Disturbance-Utilizing Control Law

2.6.1 Behavior of the Riccati/Linear System of Matric Differential Equations. The behavior of the disturbance-utilizing optimal control law Equation (2.27) depends on the values of the time-varying gain matrices $K_x(T)$, $K_{xc}(t)$ and $K_{xz}(t)$, which are the solutions of the differential Equations (2.21)-(2.23); therefore, a study of these solutions is appropriate.

Results from the "conventional" linear-quadratic regulator problem (i.e., with no disturbances present) apply to the solution of Equation (2.21) (see Reference [8]). That is, if S and Q(t) are non-negative definite, T is specified, and R(t) is positive-definite, then Equation (2.21) has $K_x(t)$ as its unique n x n positive-definite solution. This equation is completely uncoupled from the others, and so it can be solved independently for $K_x(t)$. Note that Equations (2.22) and (2.23) each depend on $K_x(t)$, exhibiting a unilateral coupling.

Since the boundary condition for each equation is given at the terminal time, t = T, the equations must be solved in backward time. By making the substitution

$$\tau = T - t \tag{2.29}$$

in Equations (2.21)-(2.23), where $(T - t_o) \geq \tau \geq 0$, the following "backward time" equations are obtained (here $(\dot{\ }) = d/d\tau$):

$$\dot{K}_x = (A - BR^{-1}B^TK_x)^TK_x + K_xA + C^TQC \quad ;K_x(0) = C^TSC \tag{2.30}$$

$$\dot{K}_{xc} = (A - BR^{-1}B^TK_x)^TK_{xc} + K_{xc}E - C^TQG \;;K_{xc}(0) = -C^TSG \tag{2.31}$$

$$\dot{K}_{xz} = (A - BR^{-1}B^TK_x)^TK_{xz} + K_xFH + K_{xz}D \;;K_{xz}(0) = 0 \tag{2.32}$$

These differential equations are readily solved numerically, using digital computer numerical integration routines, such as second-order or fourth-order Runge-Kutta, to obtain $K_x(t)$, $K_{xc}(t)$, and $K_{xz}(t)$ gain histories for the optimal control law Equation (2.27) over the total control interval $t_o \leq t \leq T$. The remaining three equations, (2.24)-(2.26), do not enter into computation of the optimal control law; but the latter two Equations, (2.25) and (2.26), along with Equation (2.23), have important effects on state space domains in which positive disturbance utilization is possible. This topic is considered in Section 2.8 of this chapter.

2.6.2 Existence of Steady-State Equilibrium Solutions of the Riccati/Linear System. The special case in which the matrices A, B, C, D, E, F, G, H, Q, R and S are all constant matrices is important in practical applications and can be analytically studied somewhat further than the time-varying case. The existence of steady-state solutions of the matric differential Equations (2.21)-(2.26) as $T \to \infty$ will now be examined for the constant case.

2.6.2.1 Existence of a Steady-State Solution $\bar{K}_x$. The standard linear-quadratic optimal control problem is concerned with the task of finding the optimal control u^o to minimize a quadratic performance index Equation (2.5) subject to an undisturbed linear state model

$$\dot{x} = Ax + Bu \tag{2.33}$$

$$y = Cx \tag{2.34}$$

The non-negative definite matrix Q in the performance index Equation (2.5) can always be expressed as [14]

$$Q = H_1^T H_1 \tag{2.35}$$

for some unique matrix H_1. If the matrix pair $[A, H_1]$ is completely observable and the matrix pair $[A, B]$ is completely controllable, then it can be shown [15] that, for the time invariant problem, as $T \to \infty$, the solution $K_x(\tau)$ of the matric Riccati Equation (2.30) is uniformly asymptotically stable to a well-defined matrix $\bar{K}_x$

$$\lim_{\tau \to \infty} K_x(\tau) = \bar{K}_x \tag{2.36}$$

where $\bar{K}_x$ is the unique, positive-definite solution of the so-called matric <u>algebraic</u> Riccati equation

$$(A - BR^{-1}B^T\bar{K}_x)^T\bar{K}_x + \bar{K}_xA + C^TQC = 0 \tag{2.37}$$

The composite system described by Equation (2.9), however, is not completely controllable, since the control u(t) has no effect on the disturbance z or the set-point/servo-command state c. Thus, the aforementioned sufficient conditions for the existence of a steady-state solution are not met by the augmented matrix $\bar{P}$ (Equation (2.19)).

An alternative approach [16] to the problem of existence is to partition the composite system equations into a completely-controllable (c.c.) part and a totally uncontrollable part. Then the conditions on the existence of steady-state value of gain will apply to the c.c. part. For this purpose, the composite matrices A and B will be re-partitioned as

$$\bar{A} = \left[\begin{array}{c|c} A_1 & A_3 \\ \hline 0 & A_2 \end{array}\right]; \quad \bar{B} = \left[\begin{array}{c} B_1 \\ \hline 0 \end{array}\right] \tag{2.38}$$

where the following identifications are made:

$$A_1 = A \tag{2.39}$$

$$A_3 = \left[\begin{array}{c|c} 0 & FH \end{array}\right] \tag{2.40}$$

$$A_2 = \left[\begin{array}{c|c} E & 0 \\ \hline 0 & D \end{array}\right] \tag{2.41}$$

$$B_1 = B \tag{2.42}$$

It is observed that Equation (2.30) is the Riccati equation for the auxiliary problem with system equation

$$\dot{x}_I = A_1 x_I + B_1 u \tag{2.43}$$

$$y = C x_I \tag{2.44}$$

where the composite vector $\tilde{x}$ has been repartitioned as

$$\tilde{x} = \left(\frac{x_I}{x_{II}}\right) \tag{2.45}$$

where

$$x_I = x$$

$$x_{II} = \left(\frac{c}{z}\right) \tag{2.46}$$

Now, if the matrix pair $[A_1, H_1]$ is completely observable and the matrix pair $[A_1, B_1]$ is completely controllable, then as $T \to \infty$ the matric Riccati differential Equation (2.30) has a solution which approaches a unique, constant, positive-definite value $\bar{K}_x$, and furthermore, $\bar{K}_x$ is the unique, positive definite solution of the associated matric algebraic equation (2.37).

The inclusion of the condition that the matrix pair $[A_1, H_1]$ be completely observable ensures that the eigenvalues of the closed-loop matrix

$$A_{CL} = (A - BR^{-1}B^T K_x) \tag{2.47}$$

will have negative real parts [15, 17; pp. 39-43], even if the original open-loop system is unstable. (The condition on $[A_1, H_1]$ has the effect of ensuring that the elements of the state vector x are all "observed" by the performance index J).

2.6.2.2 Existence of a Steady State Solution $\bar{K}_{xc}$. If the conditions for the existence of $\bar{K}_x$ are met, then as $\tau\to\infty$, Equation 2.31 may be written

$$\dot{K}_{xc}(\tau) = A_{CL}{}^T K_{xc}(\tau) + K_{xc}(\tau)\ E - C^T QG;\ K_{xc}(0) = -C^T SG \quad (2.48)$$

Equation (2.48) may be rewritten as the equivalent vector differential equation ([20], Chapter 12)

$$\dot{k}_{xc}(\tau) = A_{xc} k_{xc}(\tau) - c_{xc};\ k_{xc}(0) = k_{0xc} \quad (2.49)$$

where k_{xc} is now a $(n\nu \times 1)$ vector whose elements are the elements of the (n by ν) matrix K_{xc}. Similarly, c_{xc} is a ($n\nu$ by 1) vector whose elements are the elements of $C^T QG$, and k_0xc is the ($n\nu$ by 1) initial-condition vector whose elements are the elements of $-C^T SG$. The matrix A_{xc} is the ($n\nu$ by $n\nu$) square matrix defined by the Kronecker sum

$$A_{xc} = A_{CL} \times I_\nu + I_n \times E^T \quad (2.50)$$

which is the sum of two Kronecker products. Recall that A_{CL} is an (n by n) matrix and E is a (ν x ν) matrix.

Since Equation (2.49) is equivalent to Equation (2.48), the stability of Equation (2.48) is determined by the eigenvalues of A_{xc}. Bellman ([20] , p. 230) shows that the eigenvalues of the Kronecker sum

$$A_K = A_o XI + I_n XB_o \qquad (2.51)$$

are
$$\lambda_i(A_o) + \lambda_j(B_o);$$

$$i = 1, 2, \ldots, n ; j=1, 2, \ldots \nu \qquad (2.52)$$

where A_o is an (n by n) matrix and B_o is a (ν by ν) matrix, the eigenvalues of A_o are $\lambda_i(A_o)$ and the eigenvalues of B_o are $\lambda_j(B_o)$. Using the fact that the eigenvalues of E^T and of E are identical, we may thus determine the eigenvalues of A_{xc} by

$$\lambda_i (A_{CL}) + \lambda_j (E);$$
$$i = 1, 2, \ldots , n ; j = 1, 2, \ldots, \nu \qquad (2.53)$$

where $\lambda_i(A_{CL})$ and $\lambda_j(E)$ are the eigenvalues of A_{CL} and E, respectively. The vector differential Equation

(2.49) will be asymptotically stable if, and only if, the eigenvalues of A_{xc} have negative real parts. Therefore, we may state the following

<u>CONDITION 2.1</u> A necessary and sufficient condition that K_{xc} has a steady-state value $\bar{K}_{xc}$ is that

$$\text{Re}\left[\lambda_i(A_{CL}) + \lambda_j(E)\right] < 0;$$
$$\text{for any } i = 1, 2, \ldots, n \quad ; \; j = 1, 2, \ldots, \nu \tag{2.54}$$

i.e., that the sum of the real parts of any eigenvalue of the closed-loop system matrix A_{CL} and any eigenvalue of the set-point/command matrix E is negative.

Condition 2.1 is <u>necessary</u> because A_{xc} must be an asymptotically stable matrix (Equation 2.49 must have an asymptotically stable solution $k_{xc}(\tau)$) to ensure that $\bar{K}_{xc}$ exists. Condition 2.1 is a <u>sufficient</u> condition because, if A_{xc} is an asymptotically stable matrix, then $k_{xc}(\tau)$ is asymptotically stable and hence $\bar{K}_{xc}$ exists.

Note that Condition 2.1 will be satisfied for the constant set-point problem, in which $E \equiv 0$ since A_{CL} is an asymptotically stable matrix under the assumption that the pair [A, B] is completely controllable, and Q is non-negative definite so that

$$R_e\left[\lambda_i(A_{CL})\right] < 0; \text{ for all } i = 1, 2, \ldots, n \tag{2.55}$$

In addition, Condition 2.1 will be satisfied by a servo-command dynamic system (2.7) such that E has the property

$$\mathrm{Re}[\lambda_j(E)] \leq 0 \tag{2.56}$$

Furthermore, Condition 2.1 may be satisfied by certain E matrices having eigenvalues with <u>positive</u> real parts if it is known that A_{CL} has a certain prescribed degree of asymptotic stability such as

$$\mathrm{Re}\left[\lambda_i(A_{CL})\right] \leq -\sigma_{CL};\ \sigma_{CL} > 0 \tag{2.57}$$

where σ_{CL} is sufficiently large. In that case Condition 2.1 would be satisfied for

$$\mathrm{Re}\left[\lambda_j(E)\right] < \sigma_{CL};\ \sigma_{CL} > 0 \tag{2.58}$$

If Condition 2.1 is satisfied. then a steady-state value $\bar{K}_{xc}$ will exist, such that

$$\lim_{\tau \to \infty} K_{xc}(\tau) = \bar{K}_{xc} \tag{2.59}$$

Moreover, when the constant matrix $\bar{K}_{xc}$ is substituted in Equation (2.31), the <u>algebraic</u> equation

$$A_{CL}^T\bar{K}_{xc} + \bar{K}_{xc}E = C^TQG \tag{2.60}$$

is obtained, where A_{CL} is defined by Equation (2.47) and now depends on the constant limit $\bar{K}_x$. In a manner similar to that used in the analysis of the <u>differential</u> Equation (2.48), Equation (2.60) may be written as the equivalent vector algebraic equation [20; pp.231]

$$A_{xc}\,\bar{k}_{xc} = c_{xc} \tag{2.61}$$

where A_{xc} is defined in Equation (2.50), the vector c_{xc} was defined following Equation (2.49) and $\bar{k}_{xc}$ is the <u>constant</u> ($n\nu$ by 1) vector whose elements are the elements of the matrix $\bar{K}_{xc}$. Since the eigenvalues of A_{xc} are $\lambda_i(A_{CL}) + \lambda_j(E)$, the algebraic equation (2.61) has a unique solution if, and only if [20; pp. 231],

$$\begin{aligned} &\lambda_i(A_{CL}) + \lambda_j(E) \neq 0; \\ &\text{for any } i = 1, 2, \ldots, n \;\; ; \; j = 1, 2, \ldots, \nu \end{aligned} \tag{2.62}$$

Moreover, since Equation (2.61) is equivalent to Equation (2.60), we can state the following:

<u>COROLLARY 2.1</u> A necessary and sufficient condition that the linear matric algebraic Equation (2.61) has a unique solution $\bar{K}_{xc}$, for any C^TQG, is that

$$\lambda_i(A_{CL}) + \lambda_j(E) \neq 0 \; ;$$
$$\text{for any } i = 1, 2, \ldots, n \; ; \; j = 1, 2, \ldots, \nu \qquad (2.63)$$

which is automatically satisfied if Condition 2.1 is met.

The <u>necessary</u> part of Corollary 2.1 follows since the eigenvalues of A_{XC} must be non-zero for the existence of a unique solution $\bar{k}_{XC}$ in Equation (2.61) and hence, for the existence of matrix $\bar{K}_{XC}$. The <u>sufficient</u> part of Corollary 2.1 follows because, if the eigenvalues of A_{XC} are non-zero, it is guaranteed that a unique solution $\bar{k}_{XC}$ exists and hence that a unique solution $\bar{K}_{XC}$ exists.

A parallel approach is used in the following sections to determine the existence conditions for steady-state gains $\bar{K}_{XZ}$, $\bar{K}_C$, $\bar{K}_{CZ}$ and $\bar{K}_Z$. In each case, a linear matric equation is analyzed by examining an equivalent linear vector-matric equation.

2.6.2.3 <u>Existence of a Steady-State Solution $\bar{K}_{XZ}$</u>. The conditions for the existence of a steady-state matrix

$$\bar{K}_{XZ} = \lim_{\tau \to \infty} K_{XZ}(\tau) \qquad (2.64)$$

may be determined by an approach like that of section 2.6.2.2. The differential equation (2.32) involving

$K_{xz}(\tau)$ may be written (if $\bar{K}_x$ exists as $\tau\to\infty$) as follows:

$$\dot{K}_{xz}(\tau) = A_{CL}^T K_{xz}(\tau) + K_{xz}(\tau)D + \bar{K}_x FH \quad ; \; K_{xz}(0) = 0 \quad (2.65)$$

Equation (2.65) may be expressed by the equivalent vector differential equation

$$\dot{k}_{xz}(\tau) = A_{xz}k_{xz}(\tau) + c_{xz} \quad ; \; k_{xz}(0) = 0 \qquad (2.66)$$

where k_{xz} is an $(n\rho$ by 1) vector whose elements are the elements of the matrix K_{xz}, and c_{xz} is an $(n\rho$ by 1) vector whose elements are the elements of $\bar{K}_x FH$. A_{xz} is the $(n\rho$ by $n\rho)$ square matrix defined by the Kronecker sum

$$A_{xz} = A_{CL} X I_\rho + I_n X D^T \quad . \qquad (2.67)$$

Since D^T and D have the same eigenvalues, the eigenvalues of A_{xz} are

$$\lambda_i(A_{CL}) + \lambda_j(D) \quad ; \; i = 1, 2, \ldots, n; \quad j = 1, 2, \ldots, \rho \quad (2.68)$$

where $\lambda_i(A_{CL})$ are the eigenvalues of the closed-loop system (which have negative real parts if the pair [A, B] is completely controllable and Q is non-negative definite), and

$\lambda_j(D)$ are the eigenvalues of the system matrix of the disturbance process, Equation (2.4). The vector differential Equation (2.66) will be asymptotically stable if, and only if, the eigenvalues of A_{XZ} have negative real parts. We therefore have

<u>CONDITION 2.2</u> A necessary and sufficient condition that K_{XZ} has a steady-state value is that

$$\begin{aligned} &\text{Re}\left[\lambda_i(A_{CL}) + \lambda_j(D)\right] < 0 \quad ; \\ &\text{for any } i = 1, 2, \ldots, n \quad ; \quad j = 1, 2, \ldots, \rho \end{aligned} \tag{2.69}$$

i.e., that the sum of the real parts of any eigenvalue of the closed-loop system matrix A_{CL} and any eigenvalue of the disturbance process system matrix D is negative.

Since A_{CL} is asymptotically stable, Condition 2.2 will be satisfied by disturbance models such that

$$\text{Re}\left[\lambda_j(D)\right] \leq 0 \quad , \tag{2.70}$$

and, in the special case where it is known that A_{CL} possesses a certain prescribed degree of asymptotic stability such that Equation (2.57) holds, then Condition 2.2 will be satisfied by

$$\text{Re}\left[\lambda_j(D)\right] < \sigma_{CL} \; ; \; \sigma_{CL} > 0 \tag{2.71}$$

If Condition 2.2 is satisfied, then a steady-state solution $\bar{K}_{xz}$ will exist, and when this is substituted in Equation (2.32), the following matric algebraic equation is obtained:

$$A_{CL}{}^T\bar{K}_{xz} + \bar{K}_{xz}D = -\bar{K}_x FH \tag{2.72}$$

An equivalent vector algebraic equation may be written as

$$A_{xz}\bar{k}_{xz} = -c_{xz} \tag{2.73}$$

where A_{xz} is defined in Equation (2.67), c_{xz} was defined following Equation (2.66), and $\bar{k}_{xz}$ is the constant ($n\rho$ by 1) vector where elements are the elements of the matrix $\bar{K}_{xz}$. Since the eigenvalues of A_{xz} are $\lambda_i(A_{CL}) + \lambda_j(D)$, the algebraic Equation (2.73) has a unique solution if, and only if, [20; pp. 231],

$$\begin{aligned}&\lambda_i(A_{CL}) + \lambda_j(D) \neq 0;\\ &\text{for any } i = 1, 2, \ldots, n \;\; ; \;\; j = 1, 2, \ldots, \nu\end{aligned} \tag{2.74}$$

Moreover, since Equation (2.73) is equivalent to Equation (2.72), we can state the

<u>COROLLARY 2.2</u> A necessary and sufficient condition that the linear matric algebraic Equation (2.72) has a unique

solution $\bar{K}_{xz}$, for any matrix $-\bar{K}_x FH$, is that

$$\begin{aligned} &\lambda_i(A_{CL}) + \lambda_j(D) \neq 0; \\ &\text{for any } i = 1, 2, \ldots, n \;;\; j = 1, 2, \ldots, \rho \end{aligned} \tag{2.75}$$

which is automatically satisfied if Condition 2.1 is met.

2.6.2.4 Existence of a Steady-State Solution $\bar{K}_c$. If, as $\tau \to \infty$, a steady-state matrix $\bar{K}_{xc}$ exists, then Equation (2.24) leads to the "backward-time" equation

$$\begin{aligned} &\dot{K}_c(\tau) = E^T K_c(\tau) + K_c(\tau)E - \bar{K}_{xc}{}^T B R^{-1} B^T \bar{K}_{xc} + G^T Q G; \\ &K_c(0) = G^T S G \qquad . \end{aligned} \tag{2.76}$$

Equation (2.76) is stable if, and only if, the eigenvalues of E have negative real parts, which leads to

CONDITION 2.3 A necessary and sufficient condition that $\bar{K}_c$ exists is that [20; pg. 231]

$$\begin{aligned} &\mathrm{Re}\{\lambda_i(E) + \lambda_j(E)\} < 0; \\ &\text{for any } i = 1, 2, \ldots, \nu \;;\; j = 1, 2, \ldots, \nu \;; \\ &\text{(including } i = j) \end{aligned} \tag{2.77}$$

i.e., that the real part of every eigenvalue of E be negative.

This is a stronger condition than the condition for the existence of $\bar{K}_{xc}$. Note that $\bar{K}_c$ will not exist for the set-point regulator case, since for that problem, the set-point "command generator" model (Equation 2.7) has $E \equiv 0$, and therefore is not asymptotically stable. However, $\bar{K}_c$ will exist for such servo-command systems as may be modeled by an asymptotically stable linear system matrix E.

In those cases where Condition 2.3 is satisfied, the limit $\bar{K}_C$ may be substituted in Equation (2.76), resulting in the matric algebraic equation

$$E^T\bar{K}_C + \bar{K}_C E = \bar{K}_{XC} BR^{-1} B^T \bar{K}_{XC} - G^T QG \tag{2.78}$$

which has a unique solution $\bar{K}_C$ if, and only if, it satisfies

<u>COROLLARY 2.3</u> A necessary and sufficient condition that the linear matrix algebraic Equation (2.78) has a unique solution $\bar{K}_C$, for any matrix $\bar{K}_{XC}BR^{-1}B^T\bar{K}_{XC} - G^TQG$, is that $\bar{K}_{XC}$ exist and that [20; pg. 231]

$$\begin{aligned}&\lambda_i(E) + \lambda_j(E) \neq 0;\\ &\text{for any } i = 1, 2, \ldots, \nu \quad ; \quad j = 1, 2, \ldots, \nu \quad ;\\ &(\text{including } i = j)\end{aligned} \tag{2.79}$$

This will automatically be satisfied if Condition 2.3 is met.

2.6.2.5 <u>Existence of a Steady-State Solution</u> <u>$\bar{K}_{CZ}$</u>. If, as $\tau \to \infty$, $\bar{K}_{XC}$ and $\bar{K}_{XZ}$ exist, then Equation (2.25) leads to the "backward-time" differential equation

$$\begin{aligned}&\dot{K}_{CZ}(\tau) = E^T K_{CZ}(\tau) + K_{CZ}(\tau) D - \bar{K}_{XC}{}^T (BR^{-1}B^T\bar{K}_{XZ} - FH)\\ &K_{CZ}(0) = 0 \quad .\end{aligned} \tag{2.80}$$

The stability of Equation (2.80) is determined by $\lambda_i(E) + \lambda_j(D)$ as follows:

CONDITION 2.4 A necessary and sufficient condition that a steady-state solution $\bar{K}_{cz}$ exist is that $\bar{K}_{xc}$ and $\bar{K}_{xz}$ exist and

$$\begin{aligned}&\mathrm{Re}\left[\lambda_i(E) + \lambda_j(D)\right] < 0;\\&\text{for any } i = 1, 2, \ldots, \nu \; ; \; j = 1, 2, \ldots, \rho \quad .\end{aligned} \tag{2.81}$$

If $\bar{K}_{cz}$ exists, then, when it is substituted in Equation (2.80), the result is the matric algebraic equation

$$E^T\bar{K}_{cz} + \bar{K}_{cz}D = \bar{K}_{xc}^T(BR^{-1}B^T\bar{K}_{xz} - FH) \ , \tag{2.82}$$

which has a unique solution $\bar{K}_{cz}$ if, and only if, it satisfies

COROLLARY 2.4 A necessary and sufficient condition that the linear matric algebraic Equation (2.82) has a unique solution $\bar{K}_{cz}$, for any matrix $\bar{K}_{xc}^T(BR^{-1}B^T\bar{K}_{xz}-FH)$, is that $\bar{K}_{xc}$ and $\bar{K}_{xz}$exist and

$$\begin{aligned}&\lambda_i(E) + \lambda_j(D) \neq 0;\\&\text{for any } i = 1, 2, \ldots, \nu \; ; \; j = 1, 2, \ldots, \rho \quad .\end{aligned} \tag{2.83}$$

which will be automatically satisfied in Condition 2.4 is met.

2.6.2.6 Existence of a Steady-State Solution K_z. If $\bar{K}_{xz}$ exists as $\tau\to\infty$, then Equation (2.26) leads to

$$\dot{K}_z(\tau) = D^T K_z(\tau) + K_z(\tau) D - \bar{K}_{xz}^T B R^{-1} B^T \bar{K}_{xz} + \left[(FH)^T \bar{K}_{xz} + \bar{K}_{xz}^T FH\right];$$

$$K_z(0) = 0 \tag{2.84}$$

which is asymptotically stable if, and only if, it satisfies

CONDITION 2.5 A necessary and sufficient condition that a steady-state solution $\bar{K}_z$ exist is that $\bar{K}_{xz}$ exist and

$$\text{Re}\left[\lambda_i(D) + \lambda_j(D)\right] < 0;$$
$$\text{for any } i = 1, 2, \ldots, \rho \;;\; j = 1, 2, \ldots, \rho \;;$$
$$\text{(including } i = j) \tag{2.85}$$

i.e., that every eigenvalue of D have a negative real part. Note that this is a more restrictive condition (on the eigenvalues of D) than Condition 2.2.

If $\bar{K}_{xz}$ exists, then the following matric algebraic equation results from Equation (2.84)

$$D^T \bar{K}_z + \bar{K}_z D = \bar{K}_{xz}^T B R^{-1} B^T \bar{K}_{xz} - \left[(FH)^T \bar{K}_{xz} + \bar{K}_{xz}^T FH\right] \tag{2.86}$$

which has a unique solution $\bar{K}_z$ if, and only if, it satisfies

COROLLARY 2.5 A necessary and sufficient condition that the linear matric algebraic Equation (2.86) has a unique solution $\bar{K}_z$, for any matrix $\bar{K}_{xz}^T B R^{-1} B^T \bar{K}_{xz} - [(FH)^T \bar{K}_{xz} + \bar{K}_{xz}^T FH]$ is that

$\lambda_i(D) + \lambda_j(D) \neq 0;$

for any $i = 1, 1, \ldots, \rho$; $j = 1, 2, \ldots, \rho$; (2.87)

(including $i = j$)

which is automatically satisfied if Condition 2.5 is met.

2.6.3 The Steady-State Control Law. If the conditions for existence of $\bar{K}_x$, $\bar{K}_{xc}$ and $\bar{K}_{xz}$ are satisfied, then the steady-state control for the set-point/servo-tracking disturbance utilizing problem exists and can be expressed as

$$\bar{u}^o = -R^{-1}B^T \left[\bar{K}_x x + \bar{K}_{xc} c + \bar{K}_{xz} z \right]. \qquad (2.88)$$

Furthermore, the steady-state gains $\bar{K}_x$, $\bar{K}_{xc}$ and $\bar{K}_{xz}$ are found as the unique solutions of the matric algebraic Equations (2.37), (2.60) and (2.72), respectively.

The performance index for the steady state set-point/servo-tracking problem is

$$J = \tfrac{1}{2}\tilde{x}^T(\infty)\ \bar{S}\tilde{x}(\infty) + \tfrac{1}{2} \int_{t_o}^{\infty} \left[\tilde{x}^T(t)\bar{Q}\tilde{x}(t) + u^T(t)Ru(t)\right] dt. \qquad (2.89)$$

A finite value of J in the infinite-time problem requires that the set-point/servo-command vector $c(t) \to 0$ as $t \to \infty$. This can be seen by considering, for example, the problem of regulating the plant state x to a non-zero set-point x_{sp}, where x_{sp} is not a natural equilibrium point.

Maintaining x at x_{sp}, as $\tau \to \infty$, requires a constant value of control u in this case, causing the integral term in J (an integral over an infinite time interval) to be infinite in value. This property may be summarized by stating that the performance index J Equation (2.89) for the steady-state set-point/servo-tracking disturbance-utilizing problem will be infinite if the system matrix E of the set-point/servo-command model does not have eigenvalues with negative real parts.

Similarly, a finite value of J in the infinite-time disturbance-utilizing problem requires that the disturbance state vector $z(t) \to 0$ as $\tau \to \infty$. For example, a constant disturbance in the steady-state disturbance-utilization problem leads to a constant control u(t) as $\tau \to \infty$, which in turn leads to an infinite value of J. Thus, we may summarize this property by stating that the performance index J (Equation 2.89) for the steady-state set-point/servo-tracking disturbance-utilizing problem will be infinite if the system matrix D of the disturbance model does not have eigenvalues with negative real parts.

If the steady-state problem leads to an infinite value of J, then Equation (2.88) is no longer a rigorous expression for the optimal control, but may serve as a reasonable approximation for engineering purposes. Anderson and Moore [17; pg. 265] state that, in this case,

> The only general optimal control interpretation of these results is that they are the limiting results of the finite-time optimal servo problem case. That is, they have properties very close to the optimal systems designed for a large terminal time T.

2.6.4 Some Methods for Determining the Steady-State Equilibrium Values

2.6.4.1 Explicit Solution for $\bar{K}_x$. The solution of the algebraic Riccati equation

$$(A - BR^{-1}B^T\bar{K}_x)^T \bar{K}_x + \bar{K}_x A + C^T QC = 0 \tag{2.90}$$

may be found in explicit form as follows [9; pg. 121]. Consider the matrix

$$M = \left[\begin{array}{c|c} -A & BR^{-1}B^T \\ \hline C^T QC & A^T \end{array}\right] \tag{2.91}$$

Let T be any matrix which transforms M into its Jordan form L, so that

$$L = T^{-1} MT \tag{2.92}$$

Then, using the partitions of L and T, write

$$\left[\begin{array}{c|c} -A & BR^{-1}B^T \\ \hline C^TQC & A^T \end{array}\right] \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] = \left[\begin{array}{c|c} T_1 & T_2 \\ \hline T_3 & T_4 \end{array}\right] \left[\begin{array}{c|c} L_1 & L_2 \\ \hline 0 & L_4 \end{array}\right] \tag{2.93}$$

Then, provided T_1 is nonsingular, the matrix

$$\bar{K}_x = T_3 T_1^{-1} \tag{2.94}$$

is a solution of Equation (2.90).

Other methods of explicit solution for $\bar{K}_x$ may be found in [9] and [17].

2.6.4.2 Explicit Solutions of the Linear Matric Algebraic Equations. The linear matrix algebraic equations for $\bar{K}_{xc}$, $\bar{K}_{xz}$, $\bar{K}_c$, $\bar{K}_{cz}$, and $\bar{K}_z$ may each be expressed in the general form

$$N\bar{K} + \bar{K}P = C_K \tag{2.95}$$

The solution $\bar{K}$, if it exists, is unique and is expressed by ([20]; pg. 175, Theorem 6)

$$K = -\int_0^{\infty} e^{Nt} C_K e^{Pt} dt . \qquad (2.96)$$

This is seen by considering the equation

$$\dot{Z} = NZ + ZP , \quad Z(0) = C_K \qquad (2.97)$$

Which may be integrated between t = o and t = ∞; assuming that $\lim_{\tau \to \infty} Z(t) = o$, the result is

$$-C_K = N(\int_0^{\infty} Zds) + (\int_0^{\infty} Zds) P \qquad (2.98)$$

and it is seen that

$$-\int_0^{\infty} Zds = -\int_0^{\infty} e^{Nt} C_K e^{Pt} dt \qquad (2.99)$$

satisfies Equation (2.95).

Special Cases

Under special conditions such as $E \equiv 0$ or $D \equiv 0$, simpler expressions may be obtained. For example, if $E \equiv 0$ the equation for $\bar{K}_{xc}$ (Equation (2.60)) becomes

$$A_{CL}{}^T \bar{K}_{xc} = C^T QG \qquad (2.100)$$

and if $(A_{CL}^T)^{-1}$ exists, then

$$\bar{K}_{xc} = (A_{CL}^T)^{-1}C^TQG = \left[(A - BR^{-1}B^T\bar{K}_x)^T\right]^{-1} C^TQG \quad (2.101)$$

which can be solved by direct matrix operations. As a second example, if $D \equiv 0$, the equation for $\bar{K}_{xz}$ becomes

$$A_{CL}^T \bar{K}_{xz} = -\bar{K}_x FH \quad (2.102)$$

and if $(A_{CL}^T)^{-1}$ exists, then

$$\bar{K}_{xz} = - \left[(A - BR^{-1}B^T\bar{K}_x)^T\right]^{-1}\bar{K}_x FH \quad (2.103)$$

which, again, can be solved by direct matrix operations.

2.6.4.3 Computational Methods. One way to compute the steady-state solutions of the matric Riccati and the linear matric equations is to use a digital computer to integrate the backward-time equations until near-constant values are obtained. This approach has been successfully used with second- and fourth-order Runge-Kutta integration algorithms.

Other approaches are available for solving the algebraic matric equations, such as the singular perturbation method ([17], Chapter 15), which can lead to considerable savings in computer time in obtaining approximate solutions of matric Riccati equations.

2.7 The Concepts of Burden, Assistance and Utility in Disturbance-Utilization Control Theory

The solution Equation (2.18) of the Hamilton-Jacobi-Bellman equation involves the partitioned matrix $\bar{P}(t)$. The partitions of $\bar{P}(t)$ may be substituted in Equation (2.18) to obtain the expanded expression

$$V(x,c,z,t) = \tfrac{1}{2}(x^T K_x x + c^T K_c c + 2x^T K_{xc} c) + (x^T K_{xz} + c^T K_{cz}) z + \tfrac{1}{2} z^T K_z z \qquad (2.104)$$

The solution Equation (2.104) is the value J^o of the performance index J obtained under optimal control $u = u^o$ at the general initial conditions (x, c, z, t). The last term in Equation (2.104) is due to disturbances alone, and is equal to, or greater than, zero. Since it does nothing but increase the minimum value of J, Johnson has defined it as the "burden" $\mathcal{B}$ [7]:

$$\mathcal{B} \triangleq \tfrac{1}{2} z^T K_z z \qquad (2.105)$$

The term $(x^T K_{xz} + c^T K_{cz})z$ in Equation (2.104) is produced by interactions between the plant state x and the disturbance state z, and between the set-point state c and the disturbance state z. This term involves bilinear forms which may be negative, zero or positive at any time t. When this term becomes negative, it acts to further reduce the minimum value $J^o(x, c, z, t)$ of J in Equation (2.104); that is, negative values of this term actually provide assistance toward the objective of obtaining a minimum value of J. Therefore, Johnson has called the negative of this term [7] the "assistance" :

$$\mathscr{A} \triangleq -(x^T K_{xz} + c^T K_{cz}) z \; ;$$
$$x(t_o) = x; \; c(t_o) = c; \; z(t_o) = z \qquad (2.106)$$

The sign of the assistance in Equation (2.106) may itself be negative, in which case it has the effect of an additional <u>burden</u>.

The first term in Equation (2.104) does not involve the disturbance state z at all, and is, in fact, the minimum value of J that would be obtained when no disturbance is present. Therefore, any constructive action by the disturbance will be reflected in the difference between the V expression when the disturbance is present and that same V when the disturbance is absent. Johnson has defined this latter difference as "utility" $\mathscr{U}$ [7]:

$$\mathscr{U} \triangleq V \Big|_{w(t)\equiv 0} - V \Big|_{w(t) \not\equiv 0} \; . \qquad (2.107)$$

Thus, utility can be written as

$$\mathscr{U} = -(x^T K_{xz} + c^T K_{cz})z - \tfrac{1}{2} z^T K_z z \qquad (2.108)$$

or, symbolically,

$$\mathscr{U} = \mathscr{A} - \mathscr{B} \qquad (2.109)$$

Positive utility results when the assistance $\mathscr{A}$ is greater than the burden $\mathscr{B}$. Whether or not positive utility is achieved at any time t during the control interval $[t_O, T]$ depends on two general characteristics of a particular problem:

(1) the magnitudes and signs of the elements in the gain matrices and

(2) the instantaneous location of the (x, c, z) vector in relation to those regions of positive and negative utility as determined by (2.108).

2.8 Utility Domains in Extended State Space

The topological nature of the domains in (x, c, z) - space in which positive utility is achieved can be determined by examining the equations of the boundaries of those domains. Since the function (2.108) is continuous those boundaries are defined by collections of points in the (x, c, z) space where utility is zero. Such points separate domains of positive and negative utility, in general, and are defined by

$$\mathscr{U} = -(x^T K_{xz} + c^T K_{cz})z - \tfrac{1}{2} z^T K_z z = 0 \qquad (2.110)$$

This latter equation may be written as

$$\left[\left[x^T \middle| c^T\right] \left[\frac{K_{xz}}{K_{cz}}\right] + \tfrac{1}{2} z^T K_z \right] z = 0 \tag{2.111}$$

Solutions of Equation (2.111) may be considered with respect to two distinct conditions on the dimensions of K_{xz}, K_{cz} and K_z: the case where $\rho \leq (n + \nu)$ and the case where $\rho > (n + \nu)$:

(1) If $\rho \leq (n + \nu)$, Equation (2.111) is satisfied by z = 0 and by

$$\left[x^T \middle| c^T\right] = -\tfrac{1}{2} z^T K_z \left[\left[\frac{K_{xz}}{K_{cz}}\right]^T \left[\frac{K_{xz}}{K_{cz}}\right]\right]^{-1} \left[\frac{K_{xz}}{K_{cz}}\right]^T \; ; \; \rho \leq (n + \nu) \tag{2.112}$$

provided $\left[\frac{K_{xz}}{K_{cz}}\right]^T \left[\frac{K_{xz}}{K_{cz}}\right]$ has maximal rank. Note that

$$\operatorname{rank}\left[\frac{K_{xz}}{K_{cz}}\right]^T \left[\frac{K_{xz}}{K_{cz}}\right] = \operatorname{rank}\left[\frac{K_{xz}}{K_{cz}}\right] \tag{2.113a}$$

and that $\left[\frac{K_{xz}}{K_{cz}}\right]$ has dimensions $(n + \nu) \times \rho$; hence maximal rank for Equation (2.113a) in this case is ρ. For the zero set-point regulator problem, $c \equiv 0$, and therefore the $\mathscr{V} = 0$ boundaries are defined by the two expressions

$$z = 0$$

and

$$x^T = -\tfrac{1}{2} z^T K_z \left[K_{xz}{}^T K_{xz}\right]^{-1} K_{xz}{}^T ; \quad \rho \leq n \tag{2.113b}$$

provided K_{xz} has maximal rank (which in this case is ρ). If $\rho = n$, expression Equation (2.113b) reduces to

$$x^T = -\tfrac{1}{2} z^T K_z \left[K_{xz}\right]^{-1} \; ; \qquad \rho = n \tag{2.114}$$

(2) If $\rho > (n + \nu)$, the maximal rank of $\left[\dfrac{K_{xz}}{K_{cz}}\right]$ is $n + \nu$, and therefore, in this case $\left[\dfrac{K_{xz}}{K_{cz}}\right]^T \left[\dfrac{K_{xz}}{K_{cz}}\right]^{-1}$ does not exist, so an explicit solution for x^T is not available. However, Equation (2.111) is satisfied by $z = 0$ and by

$$z^T = -2 \left[x^T \middle| c^T \right] \left[\frac{K_{xz}}{K_{cz}}\right] K_z^{-1}; \qquad \rho > (n+\nu) \tag{2.115}$$

provided K_z^{-1} exists.

It follows from Equations (2.112) and (2.115) that, in the general case, the domains of positive utility are wedges, lying between linear subspaces in (x, c, z) space.

In the special case $n = \nu = \rho = 1$, the positive utility domain, as shown in <u>Figure 2.1</u>, lies between the plane $z = 0$ and the intersecting plane defined by Equation (2.112). Zero utility, for $n = \nu = \rho = 1$, corresponds to the points satisfying

$$z = 0 \tag{2.116}$$

$$z = -2x \frac{k_{xz}}{k_z} - 2c \frac{k_{cz}}{k_z} \tag{2.117}$$

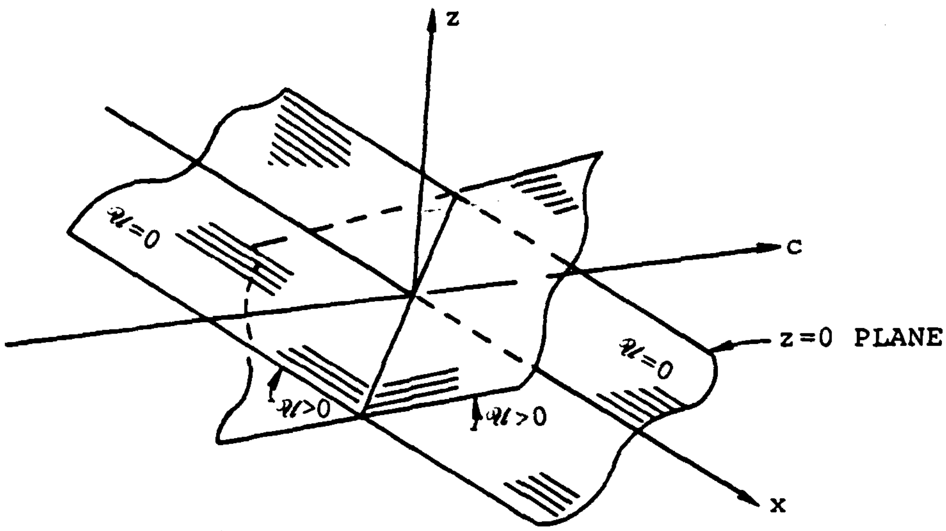

Figure 2-1. Domains of utility in (x, c, z) space for $n = \nu = \rho = 1$

which are the equations of the planes in Figure 2.1. The "tilted" plane Equation (2.117) intersects the c, z plane along the line

$$z = -2c\,\frac{k_{cz}}{k_z} \tag{2.118}$$

and intersects the x, z plane along the line

$$z = -2x\,\frac{k_{xz}}{k_z} \tag{2.119}$$

The degree of "opening" between the two planes of Equations (2.116) and (2.117) may be described in terms of two angles, Θ_{xz} and Θ_{cz}, defined by

$$\tan\theta_{xz} = \left.\frac{dz}{dx}\right|_{c=0} = -2\,\frac{k_{xz}}{k_z} \qquad (2.120)$$

$$\tan\theta_{cz} = \left.\frac{dz}{dc}\right|_{x=0} = -2\,\frac{k_{cz}}{k_z} \qquad (2.121)$$

These angles are shown in Figure 2.2.

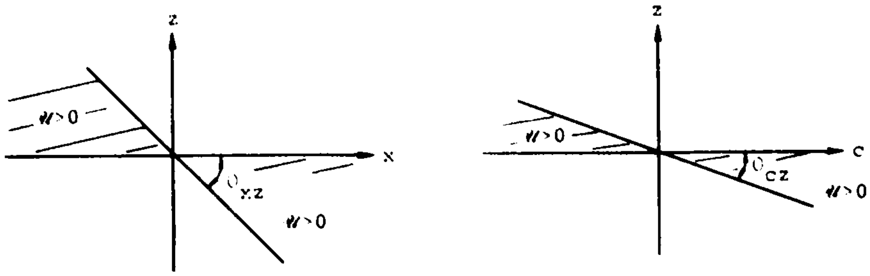

Figure 2-2. Description of positive utility domain size by Θ_{xz}, Θ_{cz}.

Since k_{xz} and k_z are never negative, Θ_{xz} is always non-positive. On the other hand, k_{cz} typically changes sign during the control interval (t_o, T), so that Θ_{cz} typically changes sign during this interval.

Description of the positive utility regions for the general case appears to be infeasible due to the associated geometrical complexities. In general, these regions of positive utility are time-varying regions, since the gains K_{xz}, K_{cz} and K_z are time-varying solutions of matric differential equations. These regions may, or may not collapse as $t \to T$, depending on the properties of the set-point/servo-command process and the disturbance process.

Some dynamic properties of the utility function are discussed in the next section.

2.9 Dynamic Properties of the Utility Function

It follows from the definition of the utility function:

$$\mathcal{U} = -(x^T K_{xz} + c^T K_{cz})\, z - \tfrac{1}{2} z^T K_z z \qquad (2.122)$$

that, if the disturbance w(t) is asymptotically stable, i.e., $z(t) \to 0$ as $t \to \infty$, one has the following

PROPERTY 2.1 If the disturbance w(t) is asymptotically stable, such that

$$\lim_{t \to \infty} z(t) = 0 \qquad (2.123)$$

then the optimal trajectory $x^o(t)$ has the property that

$$\lim_{t \to \infty} \mathcal{U}\,(x(t),\, c(t),\, z(t),\, t) = 0; \qquad (2.124)$$

for all bounded c(t).

The known terminal conditions, at t = T, on the matrices $K_{xz}(t)$, $K_{cz}(t)$ and $K_z(t)$ are

$$K_{xz}(T) = \left[0\right]_{n \times \rho} \qquad (2.125)$$

$$K_{cz}(T) = \left[0\right]_{\nu \times \rho} \qquad (2.126)$$

$$K_z(T) = \left[0\right]_{\rho \times \rho} \qquad (2.127)$$

which leads to

PROPERTY 2.2

$$\mathcal{U}(x(T),\ c(T),\ z(T),\ T) = 0 \qquad (2.128)$$

Note that the value of $\mathcal{U}$ approaches zero continuously as $t \to T$, since $K_{xz}(t)$, $K_{cz}(t)$ and $K_z(t)$ are continuous functions, each of which approaches zero as $t \to T$. The domain of positive utility for a specific problem may be very large as $t \to T$, but at the exact time $t = T$, no points in (x, c, z) space have positive utility.

For $t < T$, the value of $\mathcal{U}$ at a particular time depends on where the composite state vector lies in relation to domains of positive and negative utility. The conditions for obtaining positive utility are stated in

PROPERTY 2.3 The existence of a positive utility domain at a particular time $t_1 < T$ does not necessarily imply that the value $\mathcal{U}(x(t_1),\ c(t_1),\ z(t_1),\ t_1)$ of the utility function is positive at $t = t_1$: realization of a positive value of $\mathcal{U}$ also requires that the state vector at $t = t_1(x(t_1),\ c(t_1),\ z(t_1))$ lie in the domain of positive utility.

It is interesting to inquire if the evolution of $\mathcal{U}$ $(x(t),\ c(t),\ z(t),\ t)$ can be described by a differential equation expression. One differential equation which $\mathcal{U}$ satisfies can be derived by taking the time-derivative of $\mathcal{U}$ (Equation 2.110). Part of the result may be recognized as $-\mathcal{U}$, resulting in

$$\frac{d\mathcal{U}}{dt} + \mathcal{U} = \Phi \tag{2.129}$$

where

$$\Phi = -x^T\left[K_{xz} - K_x FH\right] z - c^T\left[K_{cz} + K_{xc}{}^T(BR^{-1}B^T K_{xz} - FH)\right] z$$
$$- \tfrac{1}{2}\, z^T\left[K_z + K_{xz}{}^T BR^{-1}B^T K_{xz} - \left[(FH)^T K_{xz} + K_{xz}{}^T FH\right]\right] z \tag{2.130}$$

The value of $d\mathcal{U}/dt$ as $t \to T$ is found by substituting $(x(T), c(T), z(T), T) = 0$ and (T) in Equation (2.129), resulting in

PROPERTY 2.4

$$\left.\frac{d\mathcal{U}}{dt}\right|_{t=T} = \left[cx(T) - Gc(T)\right]^T\left[SCFH\right]z(T)$$

$$= \left[y(T) - Y_c(T)\right]^T\left[SCFH\right]z(T) \tag{2.131}$$

$$= \left[e(T)\right]^T\left[SCFH\right]z(T)$$

where $y(T)$ is the plant output Equation (2.2) at $t = T$, $y_c(T)$ is the output Equation (2.6) of the servo-command/set-point process at $t = T$ and $e(T) = y(T) - y_c(T)$ is the servo-command (or set-point) error at $t = T$. Note that the slope of $\mathcal{U}$ at $t = T$ depends directly on the $e(T)$; if $e(T)$ is small, $d\mathcal{U}/dt$ will be near zero at $t = T$.

The value of the utility function in the steady-state (as $\tau \to \infty$) may be found, subject to the conditions of the following

PROPERTY 2.5 If the steady-state gains $\bar{K}_{xz}$, $\bar{K}_{cz}$ and $\bar{K}_z$ exist as the backward time $\tau \to \infty$, then

$$\bar{\mathscr{U}} = -(x^T\bar{K}_{xz} + c^T\bar{K}_{cz})z - \tfrac{1}{2}z^T\bar{K}_z z; \text{ for } \tau \to \infty . \qquad (2.132)$$

Thus, the value of utility in the steady-state problem may be positive, negative, or zero, depending on where the trajectory lies in (x, c, z) space relative to the steady-state domains of positive and negative utility.

It is also possible to obtain an expression for the time-derivative of utility for the steady-state problem, subject to the existence of certain steady-state gains. Equation (2.129) may be evaluated under steady-state conditions described in the following

PROPERTY 2.6 If the steady-state gains $\bar{K}_x$, $\bar{K}_{xz}$, $\bar{K}_{cz}$ and $\bar{K}_z$ exist as backward time $\tau \to \infty$, then

$$\begin{aligned}\overline{\frac{d\mathscr{U}}{dt}} = &-x^T\left[\bar{K}_{xz}D + (A - BR^{-1}B^T\bar{K}_x)^T\bar{K}_{xz}\right]z \\ &- c^T\left[\bar{K}_{cz}D + E^T\bar{K}_{cz}\right]z \\ &- \tfrac{1}{2}z^T\left[\bar{K}_z D + D^T\bar{K}_z\right]z \quad ; \\ &\text{for } \tau \to \infty \quad .\end{aligned} \qquad (2.133)$$

Thus, the time-derivative of utility in the steady-state problem may be positive, zero, or negative, depending on how the matrices in Equation (2.133) are structured and where the trajectory is located in (x, c, z) space.

<u>The Concept of Maximum Utility</u>. The condition for achieving maximum $\mathscr{U}$ with respect to the disturbance state z may be determined by taking the gradient of $\mathscr{U}$ (Equation 2.110) with respect to z:

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$$\nabla_z \mathscr{U} = -x^T K_{xz} - c^T K_{cz} - z^T K_z \tag{2.134}$$

The second gradient of $\mathscr{U}$ with respect to z is

$$\nabla_z \left[\nabla_z \mathscr{U}\right] = -K_z$$

which is negative-definite, since K_z is positive definite. Therefore we may state

<u>PROPERTY 2.7</u> The critical point condition for a maximum of $\mathscr{U}$ with respect to z is

$$K_{xz}{}^T x + K_{cz}{}^T c + K_z z = 0 \tag{2.135}$$

Equation (2.135) describes how the plant state x should be related to the disturbance state z and the set-point/servo-command state c to achieve maximum utility $\mathscr{U}$ at each time t $\epsilon[t_o, T]$.

<u>Additional Properties of $\mathscr{U}$</u>. Certain additional properties of the utility function $\mathscr{U}$ seem intuitively to be true. For example, it seems that for a certain specified terminal state-weighting matrix S, that the value of the utility function $\mathscr{U}$ at each time t $\epsilon[t_o, T]$ would depend on the relationship between Q and R in the performance index J, with $Q \equiv 0$ leading to higher values of $\mathscr{U}$.

Some numerical examples which show the relations between specified control objectives (such as utilization of disturbances and set-point regulation) and the values of S, Q and R are presented in Chapter III.

CHAPTER III

SOME SPECIAL CASES OF DISTURBANCE-UTILIZING CONTROL

3.1 Summary of Chapter III

This chapter describes the application of disturbance-utilizing control techniques to several specific examples. In particular, the following are considered: (a) A scalar regulator with a constant disturbance, in which both zero set-point and non-zero set-point operation are considered: (b) A scalar regulator with an exponentially-decaying disturbance, with zero set-point and non-zero set-point. (c) The zero set-point regulation of a second-order plant with a vector (two-dimensional) disturbance. Expressions are obtained for the limiting values (or the limiting time-derivatives when no limiting value exists) of the gains Kx, K_{xc}, K_{xz}, K_c, K_{cz} and K_z for cases (a), (b) and (c). Corresponding expressions for systems beyond the second order present formidable computation difficulties.

These examples demonstrate several of the characteristic properties of disturbance-utilizing controllers discussed in Chapter II, including positive-utility conditions, the existence of steady-state values of system gains, the impact of disturbance waveform shape on the effectiveness of disturbance-utilizing control, and the potential for significant control energy savings.

3.2 Scalar Regulator with a Piecewise-Constant Scalar Disturbance.

3.2.1 General Results. Consider the scalar system*

$$\dot{x} = x + u + w(t) \tag{3.1}$$

$$y = x \tag{3.2}$$

where w(t) is a piecewise constant scalar disturbance which is modeled by the system

$$w = Hz = z \tag{3.3}$$

$$\dot{z} = Dz + \sigma(t) = \sigma(t) \tag{3.4}$$

where H = 1, D = 0, and σ (t) is a sparse sequence of completely unknown impulses. The primary control objective is set-point regulation to a specified (given) state x_{sp}. The secondary control objective is to accomplish the primary objective while efficiently utilizing any disturbance effects which may be available. The two objectives can be achieved, using linear-quadratic theory, by minimization of the quadratic performance index

*Scalar systems of the more general form $\dot{x} = a\,x + u + w$, a = constant, can be put in the normalized form, Equation (3.1) above by introducing the time-scaling $t \rightarrow \tau/\alpha$; $\alpha = a$, and setting $\tilde{\mu} = \mu/a$.

$$J[u] = \frac{1}{2}\, s\, e_{sp}^{2}(T) + \frac{1}{2}\int_{t_o}^{T} [q\, e_{sp}^{2}(t) + r\, u^{2}(t)]dt \tag{3.5}$$

with respect to u, where $s > o$, $q \geq o$, $r > o$ and $e_{sp} \triangleq x_{sp} - x(t)$.

The family of piecewise-constant set points $[x_{sp}]$ are modeled by the fictitious dynamical system

$$x_{sp} = G\, c = c \tag{3.6}$$

$$\dot{c} = E\, c + \mu(t) = \mu(t) \tag{3.7}$$

where $G = 1$, $E = 0$, c is the set-point "state" and $\mu(t)$ is a sparse unknown sequence of impulses.

This problem is solved by applying the theory discussed in Sections 2.4 and 2.5 of Chapter II. An augmented vector, (Equation 2.8), with elements x, c and z, leads to an augmented system (Equation 2.9) and a performance index (Equation 2.10) which is equivalent to Equation (3.5). The minimization of the performance index J subject to the augmented system Equation (2.9) is accomplished by the Hamilton-Jacobi theory. The solution of the Hamilton-Jacobi-Bellman equation is expressed as the symmetric non-negative definite quadratic form

$$V(\tilde{x}, t) = \frac{1}{2}\, \tilde{x}^{T}\bar{P}(t)\, \tilde{x} \tag{3.8}$$

where $\tilde{x}$ is the augmented vector and $\bar{P}$ (t) is chosen to satisfy the Hamilton-Jacobi-Bellman Equation (2.12). $\bar{P}$ can be written in the form of a 3x3 matrix for the problem at hand as follows:

$$\bar{P} = \begin{bmatrix} k_x(t) & k_{xc}(t) & k_{xz}(t) \\ k_{xc}^T(t) & k_c(t) & k_{cz}(t) \\ k_{xz}^T(t) & k_{cz}^T(t) & k_z(t) \end{bmatrix};$$

$$\bar{P}(T) = \bar{S} \;;\; \bar{S} = \bar{C}^{\,T} S\bar{C};$$

$$\bar{C} = \left[- C \,|\, G \,|\, 0\right] \tag{3.9}$$

Substitution of Equations (3.8) and (3.9) into the Hamilton-Jacobi-Bellman Equation (2.12) results in the following set of six unilaterally-coupled differential equations:

$$\dot{k}_x = (-1 + \frac{1}{r} k_x)k_x - k_x - q \;;\quad k_x(T) = s \tag{3.10}$$

$$\dot{k}_{xc} = (-1 + \frac{1}{r} k_x)\, k_{xc} + q \;;\quad k_{xc}(T) = -s \tag{3.11}$$

$$\dot{k}_{xz} = (-1 + \frac{1}{r} k_x)\, k_{xz} - k_x \;;\quad k_{xz}(T) = 0 \tag{3.12}$$

$$\dot{k}_c = \frac{1}{r} k_{xc}^2 - q \;;\quad k_c\,(T) = s \tag{3.13}$$

$$\dot{k}_{cz} = k_{xc}(\frac{1}{r} k_{xz} - 1) \ ; \quad k_{cz}(T) = 0 \tag{3.14}$$

$$\dot{k}_{z} = \frac{1}{r} k_{xz}^2 - 2k_{xz} \ ; \quad k_z(T) = 0 \tag{3.15}$$

The six scalar equations (3.10) - (3.15) correspond to the general matric Equations (2.21) - (2.26), with $A = 1$, $B = 1$, $R = r$, $C = 1$, $Q = q$, $S = s$, $E = 0$, $G = 1$, $F = 1$, $H = 1$ and $D = 0$.

The optimal control, Equation (2.17), may be written in terms of the solution V of the Hamilton-Jacobi-Bellman equation. Thus, using the relation $\nabla_{\tilde{x}} V = \bar{P}\,\tilde{x}$ in Equation (2.17), the optimal control for the present problem may be written as

$$u^o = -\frac{b}{r}\,(k_x x + k_{xc} c + k_{xz} z) \tag{3.16}$$

The solutions of Equations (3.10) - (3.15) which must be determined over the interval $t_o \leq t \leq T$ are obtained by backward -time integration of Equations (3.10) - (3.15) "starting" with the known end conditions at $t = T$.

The <u>backward-time</u> equations associated with Equations (3.10) - (3.15) are obtained by the substitution $t = T - \tau$, where $\tau \geq 0$ is "backward time." This latter substitution transforms Equations (3.10) - (3.15) to the form (note that $\dot{(\ )} = d\,/d\tau$ in (3.17) - (3.22)):

$$\dot{k}_x(\tau) = \left(1 - \frac{1}{r}k_x(\tau)\right) k_x(\tau) + k_x(\tau) + q \; ; \quad k_x(0) = s \qquad (3.17)$$

$$\dot{k}_{xc}(\tau) = \left(1 - \frac{1}{r}k_x(\tau)\right) k_{xc}(\tau) - q \; ; \quad k_{xc}(0) = - s \qquad (3.18)$$

$$\dot{k}_{xz}(\tau) = \left(1 - \frac{1}{r}k_x(\tau)\right) k_{xz}(\tau) + k_x(\tau) \; ; \quad k_{xz}(0) = 0 \qquad (3.19)$$

$$\dot{k}_c(\tau) = - \frac{1}{r} k_{xc}^2 (\tau) + q \; ; \quad k_c(0) = s \qquad (3.20)$$

$$\dot{k}_{cz}(\tau) = k_{xc}(\tau)\left(1 - \frac{1}{r} k_{xz}(\tau)\right) \; ; \quad k_{cz}(0) = 0 \qquad (3.21)$$

$$\dot{k}_z(\tau) = -\frac{1}{r} k_{xz}^2(\tau) + 2k_{xz}(\tau) \; ; \quad k_z(0) = 0 \qquad (3.22)$$

The required solutions of Equations (3.10) - (3.15) are now found by solving Equations (3.17) - (3.22) over the positive interval $0 \leq \tau \leq (T-t_o)$ using the "initial-condition "data at $\tau = 0$.

3.2.1.1 <u>Consideration of Limiting Values as $T \rightarrow \infty$</u>

It is readily verified that (3.1) is completely controllable. Therefore, since $q \geq 0$, it is assured [15] that Equation 3.17 has a solution $k_x(\tau)$ which asymptotically approaches a certain positive value $\bar{k}_x$ as $\tau \rightarrow \infty$. That is, the limit

$$\lim_{\tau \to \infty} k_x(\tau) = \bar{k}_x \;,\; \bar{k} = \text{constant} \geq 0, \tag{3.23}$$

will exist, where $\bar{k}_x$ is the unique, positive definite solution of the "steady-state Riccati equation"

$$(1 - \frac{1}{r}\,\bar{k}_x)\;\bar{k}_x + \bar{k}_x + q = 0 \quad . \tag{3.24}$$

The positive solution of Equation (3.24) is readily found to be

$$\bar{k}_x = r\;(\;1 + \sqrt{q/r + 1})\;. \tag{3.25}$$

For the special case where $q = 0$, Equation (3.25) yields

$$\bar{k}_x = 2r\;,\quad q = 0 \tag{3.26}$$

The result of Equation (3.25), may be verified by starting with the general solution of the Riccati differential Equation (3.17) and evaluating $k_x(\tau)$ as $\tau \to \infty$. The general solution of Equation (3.17) is given by [8; pg. 777]

$$k_x(\tau) = r\,\frac{\beta + 1 + (\beta - 1)\left(\dfrac{\frac{s}{r} - 1 - \beta}{\frac{s}{r} - 1 + \beta}\right)e^{-2\beta\tau}}{1 - \left(\dfrac{\frac{s}{r} - 1 - \beta}{\frac{s}{r} - 1 + \beta}\right)e^{-2\beta\tau}} \tag{3.27}$$

where $\tau = T - t$, and for our case, $\beta = +\sqrt{q/r + 1} > 0$. Letting $\tau \to \infty$ in Equation (3.27) yields the limiting solution

$$\lim_{\tau \to \infty} k_x(\tau) = r \left(1 + \sqrt{q/r + 1} \right) \tag{3.28}$$

which is identical to the solution Equation (3.25) of the algebraic Riccati equation. Note that Equation (3.27) has other solutions that do not satisfy Equation (3.28).

The conditions for the existence of steady-state values of k_{xc}, k_{xz}, k_c, k_{cz} and k_z (Conditions 2.1 - 2.5 in Chapter II) may now be applied to the last five equations, (3.18) - (3.22).

Condition 2.1 indicates that the solution $k_{xc}(\tau)$ of Equation (3.18) asymptotically approaches a unique steady-state value $\bar{k}_{xc}$ if, and only if,

$$1 - \frac{1}{r} \bar{k}_x < 0 \; . \tag{3.29}$$

However, satisfaction of Equation (3.29) is guaranteed for this example because substitution of Equation (3.25) into Equation (3.29) yields

$$1 - \frac{1}{r} \bar{k}_x = - \sqrt{q/r + 1} < 0 \; . \tag{3.30}$$

The unique solution of Equation (3.18) is easily determined to be

$$k_{xc}(\tau) = \frac{q}{(1 - \frac{1}{r}\bar{k}_x)} + C_{xc}e^{(1 - \frac{1}{r}\bar{k}_x)\tau} \tag{3.31}$$

where

$$C_{xc} = -s - \frac{q}{(1 - \frac{1}{r}\bar{k}_x)} \quad . \tag{3.32}$$

Thus, the limiting value $\bar{k}_{xc}$ is obtained from Equation (3.31) as

$$\bar{k}_{xc} = \lim_{\tau \to \infty} k_{xc}(\tau) = \frac{q}{(1 - \frac{1}{r}\bar{k}_x)} \quad . \tag{3.33}$$

Expression Equation (3.25) for $\bar{k}_x$ may be substituted into Equation (3.33) to obtain

$$\bar{k}_{xc} = \frac{-q}{\sqrt{q/r + 1}} \quad . \tag{3.34}$$

For the special case of q = 0, (3.34) reduces to

$$\bar{k}_{xc} = 0 \quad ; \qquad q = 0 \quad . \tag{3.35}$$

Application of Condition 2.2 of Chapter II shows that a well-defined steady-state value $\bar{k}_{xz}$ will exist for this example problem, because the disturbance model has D = 0, and $\bar{k}_x$ exists. The unique solution of Equation 3.19 is found to be

$$k_{xz}(\tau) = \frac{-\bar{k}_x}{(1 - \frac{1}{r}\bar{k}_x)} + C_{xz}e^{(1-\frac{1}{r}\bar{k}_x)\tau} \tag{3.36}$$

where

$$C_{xz} = \frac{\bar{k}_x}{(1 - \frac{1}{r}\bar{k}_x)} \quad . \tag{3.37}$$

Letting $\tau\to\infty$ in Equation (3.36), the steady-state value $\bar{k}_{xz}$ is determined to be

$$\bar{k}_{xz} = \lim_{\tau \to \infty} k_{xz}(\tau) = \frac{r\ (1 + \sqrt{q/r + 1})}{\sqrt{q/r + 1}} . \tag{3.38}$$

The special case q = 0 reduces (3.38) to

$$\bar{k}_{xz} = 2r \quad ; \qquad q = 0 \quad . \tag{3.39}$$

Condition 2.3 of Chapter II indicates that existence of a steady-state value $\bar{k}_c$ of $k_c(\tau)$ requires that every eigenvalue of the matrix E in the set-point dynamic model

Equation (3.7) have a negative real part. This condition is not satisfied for this example (see Equation (3.7)). as we have $E = 0$. Thus, for this example, no steady-state value $\bar{k}_c$ of $k_c(\tau)$ exists, in general. We may, however, find the limiting steady-state slope $\bar{\dot{k}}_c$ of the solution $k_c(\tau)$ as $\tau \to \infty$. For this purpose, the steady-state value of $\kappa_{xc}(\tau)$ may be substituted into Equation (3.20) to obtain

$$\bar{\dot{k}}_c = \lim_{\tau \to \infty} \dot{k}_c(\tau) = \frac{q^2}{q + r} \tag{3.40}$$

which shows that in backward-time τ the waveform of $k_c(\tau)$ approaches a positive-slope ramp (a negative-slope ramp in forward time). Although Equation (3.20) does not have an asymptotically stable steady-state solution in general, the special case $q = 0$ does lead to the condition that the limiting rate Equation (3.40) is zero. The corresponding value of $\lim_{\tau\to\infty} \kappa_c(\tau)$ in that case must be determined by integration.

The solution $k_{cz}(\tau)$ of Equation (3.21) has no asymptotically stable steady-state solution because E and D are both zero in this particular example. However, the limiting slope $\bar{\dot{k}}_{cz}(\tau)$ as $\tau\to\infty$ is found by substituting the values of $\bar{k}_{xc}$ and $\bar{k}_{xz}$ into Equation (3.21) to obtain

$$\bar{\dot{k}}_{cz} = \lim_{\tau \to \infty} \dot{k}_{cz}(\tau) = \frac{r\;q}{q + r} > 0. \tag{3.41}$$

It may be observed from Equation (3.41) that the special condition $q = 0$ yields a limiting rate which is zero. The corresponding value $\lim\limits_{\tau \to \infty} k_{cz}(\tau)$ must be determined by integration.

An asymptotically stable limiting value $\bar{k}_z$ for the solution $k_z(\tau)$ does not exist for Equation (3.22) because $D = 0$ in this example. As before, the limiting rate $\bar{\dot{k}}_z(\tau)$ as $\tau \to \infty$ can be calculated rather easily. Namely, if $\bar{k}_{xz}$ is substituted into Equation (3.22) one obtains

$$\bar{\dot{k}}_z = \lim_{\tau \to \infty} \dot{k}_z(\tau) = \frac{-r\left(1 + \sqrt{q/r + 1}\right)^2}{q + r} + 2\,\frac{r\left(1 + \sqrt{q/r + 1}\right)}{\sqrt{q/r + 1}} \tag{3.42}$$

As before, the special case $q = 0$ results in a zero value of the limiting rate Equation (3.42); however, more generally, Equation (3.42) describes a positive-slope ramp (a negative-slope ramp in forward time) which, of course, is unbounded as $\tau \to \infty$.

Table 3-1 summarizes the limiting steady-state gains for this example with piecewise-constant disturbances.

Figure 3-1 summarizes the waveform properties of the forward-time gain histories in this example, for general non-negative values of s, q and r.

Only the first three gains $k_x(t)$, $k_{xc}(t)$ and $k_{xz}(t)$ are required in computing the optimal control $u^\circ(t)$ for this example (see Equation (3.16)); however, the two "gains" $k_{cz}(t)$ and $k_z(t)$, along with $k_{xz}(t)$ are useful in computing the utility function Ψ.

The utility function Ψ for this example is found from Equation (2-108) to be

$$\Psi = -k_{xz}xz - k_{cz}cz - \tfrac{1}{2} k_z z^2. \qquad (3.43)$$

The positive-utility domains for examples of this type were determined in Chapter II as the "opening" between two planes, as shown in Figure 2-1. The degree of "opening" between the planes was expressed in terms of two angles, defined by

$$\tan \theta_{xz} = -2 \frac{k_{xz}}{k_z} \qquad (3.44)$$

$$\tan \theta_{cz} = -2 \frac{k_{cz}}{k_z} \qquad (3.45)$$

TABLE 3-1. STEADY-STATE GAINS FOR SCALAR SET-POINT REGULATOR WITH CONSTANT DISTURBANCE

	For $q \neq 0$, $r \neq 0$,	For $q = 0$, $r \neq 0$
$\bar{k}_x$	$r(1 + \sqrt{q^2 r + 1})$	$2r$
$\bar{k}_{xc}$	$\dfrac{-q}{\sqrt{q^2 r + 1}}$	0
$\bar{k}_{xz}$	$\dfrac{r(1 + \sqrt{q^2 r + 1})}{\sqrt{q^2 r + 1}}$	$2r$
$\bar{k}_c$	no limit exists	0
$\bar{k}_{cz}$	no limit exists	0
$\bar{k}_z$	no limit exists	$2r$

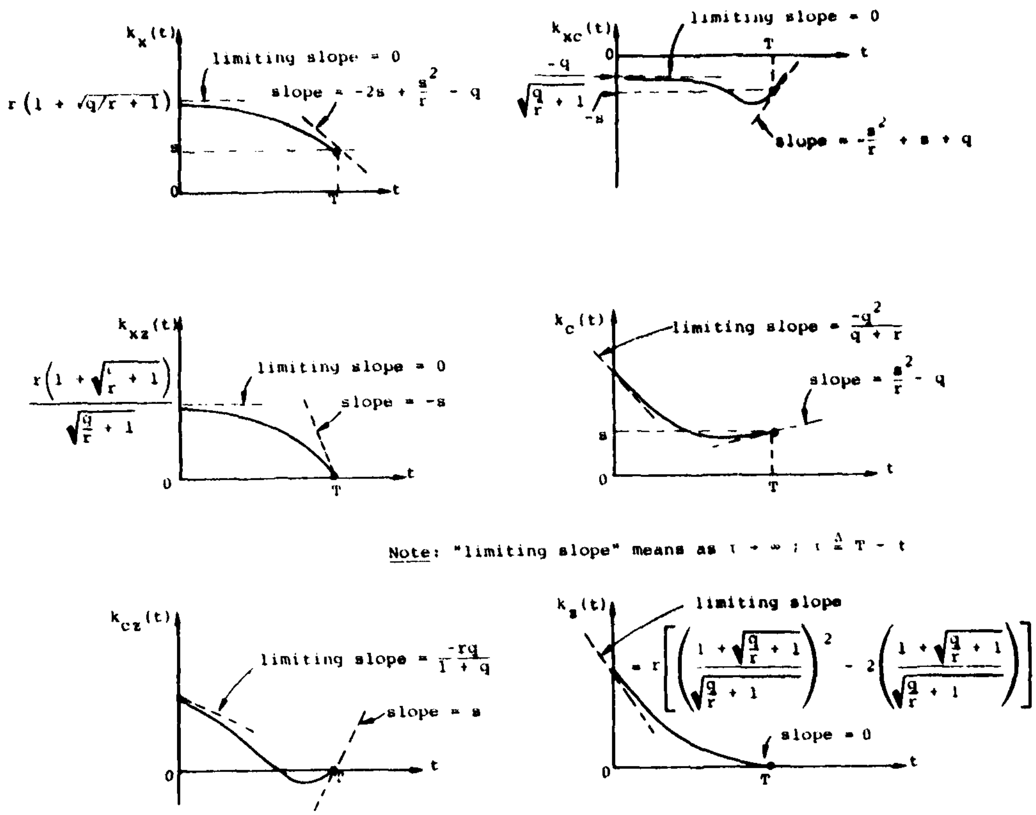

Figure 3-1. Typical forward-time gain histories for a scalar set-point regulator with a constant disturbance and disturbance-utilizing control.

which are illustrated in Figure 2-2. In general, non-negative values of q and r result in a bounded steady-state value of k_{xz} but an unbounded value of k_z as $\tau \to \infty$ (see Figure 3-1); hence, in the steady-state;

$$\bar{\theta}_{xz} = \lim_{\tau \to \infty} \theta_{xz}(\tau) = 0 \text{ , general non-negative values of } r, q \quad . \tag{3.46}$$

An interesting special case for this example occurs for the value $q = 0$, causing the right side of Equation (3.42) to then be zero, and giving the non-zero steady-state value of Equation (3.46) as

$$\bar{\theta}_{xz} = \lim_{\tau \to \infty} \theta_{xz}(\tau) = \tan^{-1}(-2) = -63.43^{\circ}, \; q = 0. \tag{3.47}$$

Equation (3.44) is indeterminant at $\tau = 0$ (i.e., at the terminal time $t = T$); however, by applying L'Hospital's rule to the ratio

$$\frac{\dot{k}_{xz}}{\dot{k}_z} = \frac{(1 - \frac{1}{r} k_x) \, k_{xz} + k_x}{-\frac{1}{r} k_{xz}^{\;2} + 2 \, k_{xz}} \tag{3.48}$$

it is easily determined that

$$\lim_{\tau \to 0} \left(\frac{d\ k_{xz}}{d\ k_z} \right) = \frac{s}{0} \tag{3.49}$$

and therefore, if $s > 0$, Equation (3.44) yields

$$\bar{\theta}_{xz} = \lim_{\tau \to 0} \theta_{xz} = -90^{\circ}; \quad s > 0. \tag{3.50}$$

In a similar manner, Equation (3.45) yields, in general, the indeterminant

$$\lim_{\tau \to \infty} \tan \theta_{cz}(\tau) = -2\left(\frac{k_{cz}}{k_z}\right) = -2\left(\frac{\infty}{\infty}\right) \tag{3.51}$$

which, by application of L'Hospital's rule gives

$$\bar{\theta}_{cz} = \lim_{\tau \to \infty} \theta_{cz}(\tau) = \frac{-2\ \dfrac{q}{1+q}}{-\left[\dfrac{1+\sqrt{q/r+1}}{\sqrt{q/r+1}}\right]^2 + 2\left[\dfrac{1+\sqrt{q/r+1}}{\sqrt{q/r+1}}\right]} \tag{3.52}$$

The denominator of Equation (3.52) is positive for all positive r and q. When $q = 0$, the denominator is zero, in which case Equation (3.51) yields the limiting value $\bar{\theta}_{cz} = 0$. Therefore, the range of variations of $\bar{\theta}_{cz}$ is, in general,

$$-90^{\circ} < \bar{\theta}_{cz} \leq 0. \tag{3.53}$$

The value of θ_{cz} at $\tau = 0$ (i.e., at terminal time $t = T$) is found by applying L'Hospital's rule to obtain

$$\lim_{\tau\to 0} \tan \theta_{cz} = -2 \lim_{\tau\to 0}\left(\frac{\dot{k}_{cz}}{\dot{k}_z}\right) = \frac{2\ s}{0} \tag{3.54}$$

which, for $s > 0$, leads to the value

$$\theta_{cz}(t=T) = \lim_{\tau\to 0} \theta_{cz}\ (\tau) = +90^{\circ}\ . \tag{3.55}$$

It is interesting to note that $\theta_{cz}(t)$ may change sign over the interval t in $[t_o, T]$.

The foregoing properties of $\theta_{xz}(t)$ and $\theta_{cz}(t)$ are summarized in Table 3-2, for several combinations of q, r and τ. Some typical domains of positive utility for this example are shown in Figures 3-2 and 3-3, for the case of large values of τ (near steady-state) and small values of τ (near terminal-time), respectively.

3.2.2 Some Specific Numerical Examples

3.2.2.1 Zero Set-Point Regulator; Scalar Plant, Constant Disturbance.

Computer results for these numerical examples were obtained from a CDC-6600 program using Runge-Kutta fourth-order integration with a computation interval of 0.02 seconds for integrating the plant equations and the gain equations. The ideal case of direct on-line

TABLE 3-2. PROPERTIES OF THE UTILITY DOMAIN "OPENING" ANGLES $\theta_{xz}(t)$ AND $\theta_{cz}(t)$ FOR A SCALAR SET-POINT REGULATOR WITH CONSTANT DISTURBANCE

θ_{xz} (Degrees) for $t \to \infty$ (steady-state)	q	r
00.00	$q > 0$	$r > 0$
-63.43	0	$r > 0$

θ_{xz} (Degrees) for $t \to 0$ (at terminal time)	q	r
-90.00	$q \geq 0$	$r > 0$

θ_{cz} (Degrees) for $t \to \infty$ (steady-state)	q	r
-63.43	1	1
-63.43	10	1
-84.81	1	10
-89.43	1	100
00.00	0	$r > 0$

θ_{cz} (Degrees) for $t \to 0$ (at terminal time)	q	r
+90.00	$q \geq 0$	$r > 0$

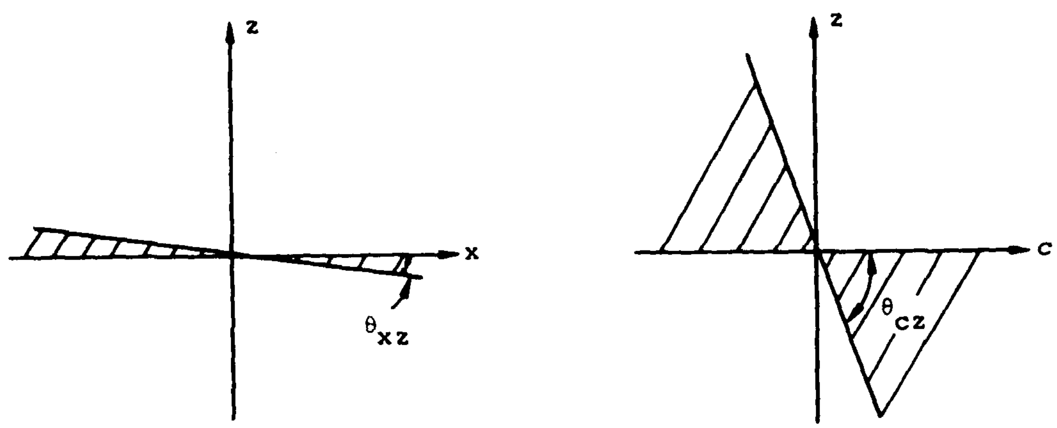

Figure 3-2. Typical domains of positive utility in x-z and c-z state space for large values of backward time ($\tau \to \infty$, near steady-state); scalar plant set-point regulator, constant disturbance, $q > 0$ and $r > o$.

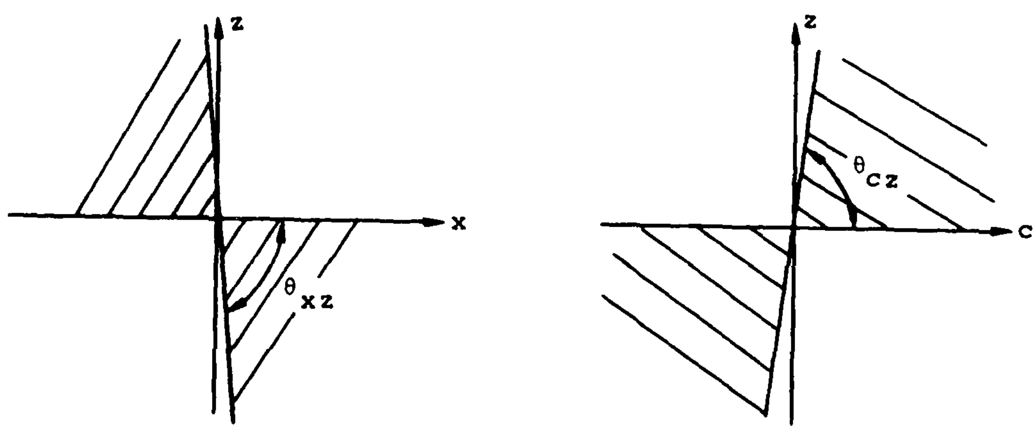

Figure 3-3. Typical domains of positive utility in x-z and c-z state space for small values of backward time (near the terminal time); scalar plant, set-point regulator, constant disturbance, $q > 0$ and $r > 0$.

measurements of the entire plant state x and the entire disturbance state z was assumed for these studies. This assumption permitted determination of the performance of the disturbance-utilizing controller without regard to the imperfections introduced by the state-reconstructors.

Numerical solutions of the three differential equations ((3-10), (3-12) and (3-15)) are presented in Figure 3-4 for the case $s = 1$, $q = 1$, $r = 1$, $T = 6.0$. These specific results may be compared with the typical histories sketched in Figure 3-1.

3.2.2.1.1 Negative Disturbance, Positive x(o).

Numerical results for the zero set-point regulator with $s = 1$, $q = o$, $r = 1$, $T = 6.0$, $w(t) = -16.1$ and $x(o) = 30.0$ are shown in Figures 3-5 and 3-6. Although the structure of this particular example results in the existence of steady-state values for the three scalar gains k_x, k_{xz} and k_z when $q = o$ (see Figure 3-5) this outcome is not typical for the general case. It may be seen that the x-z state trajectory (Figure 3-6) remains inside the positive utility sector for each $t \in [t_o, T]$ and thus the value of the utility function is positive for the whole control interval, except at $t = T$. The control u° and state x for the corresponding linear-quadratic (LQ) controller are shown as dashed curves in Figure 3-6 for comparison. All parameter values for the LQ controller are the same as those of the disturbance-utilizing (DUC) case, except that the LQ example does not

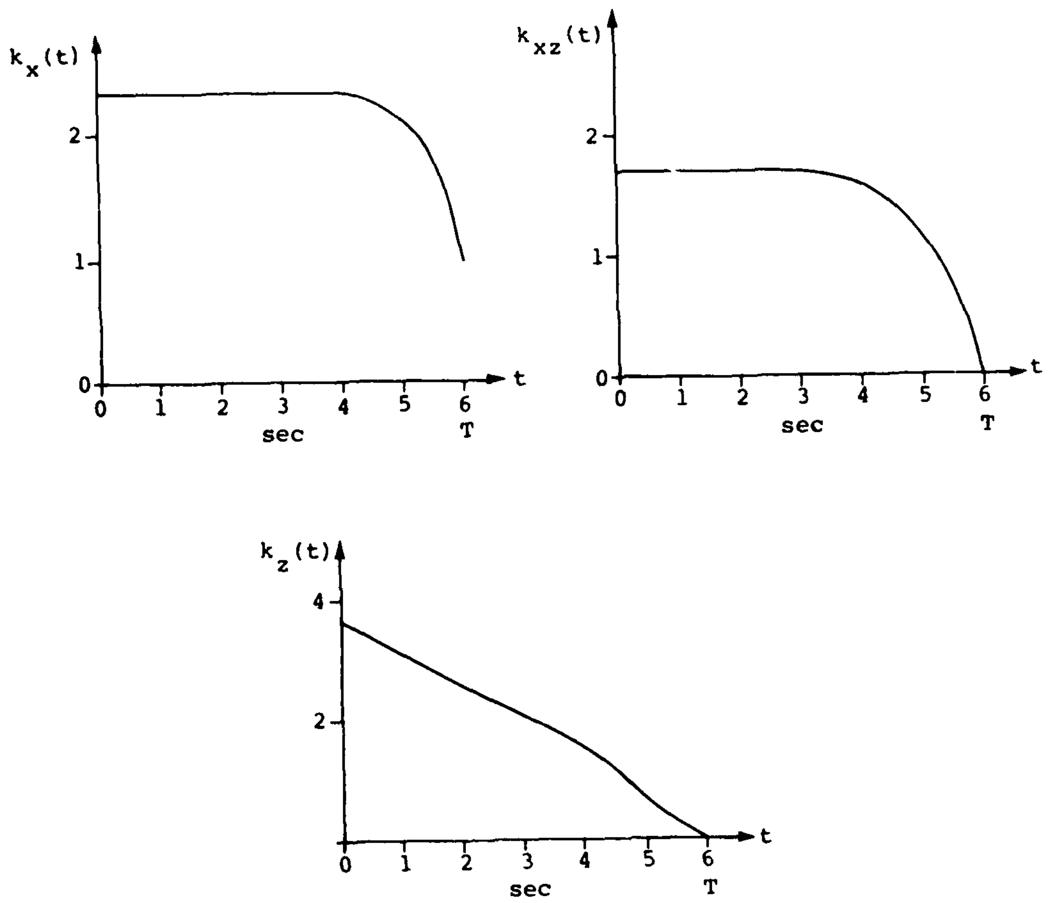

Figure 3-4. Numerical gain solutions for case of: scalar plant, scalar disturbance, zero set-point, $s = 1$, $q = 1$, $r = 1$ and $T = 6.0$.

utilize or otherwise account for the disturbance. Thus, the LQ example uses the conventional LQ control $u^{\circ}_{LQ}(t) = -\frac{b^2}{r} k_x(t)$, where $k_x(t)$ is precisely the <u>same</u> time function as $k_x(t)$ in <u>Figure 3-5</u>. The bias effects of the constant disturbance are readily seen in the LQ control and state histories (<u>Figure 3-6</u>); these effects do not appear in the disturbance/utilizing controller, however. <u>Table 3-3</u> summarizes the terminal-time performance of this example in terms of the components J_m, J_q and J_r of the performance index J, the terminal-state regulation error ϵ, the value of the control energy EU consumed during the control interval [0, T] and the "effectiveness" parameters δ_T, δ_M and δ_E; the pertinent parameter definitions are as follows:

$$J_m = \tfrac{1}{2}\, s\, e^2\,(T) \tag{3.56}$$

$$J_q = \tfrac{1}{2}\int_0^T q\, x^2\,(t)\, dt \tag{3.57}$$

$$J_r = \tfrac{1}{2}\int_0^T r\,[u^{\circ}(t)]^2 dt \tag{3.58}$$

$$EU = \tfrac{1}{2}\int_0^T [u^{\circ}(t)]^2 dt \tag{3.59}$$

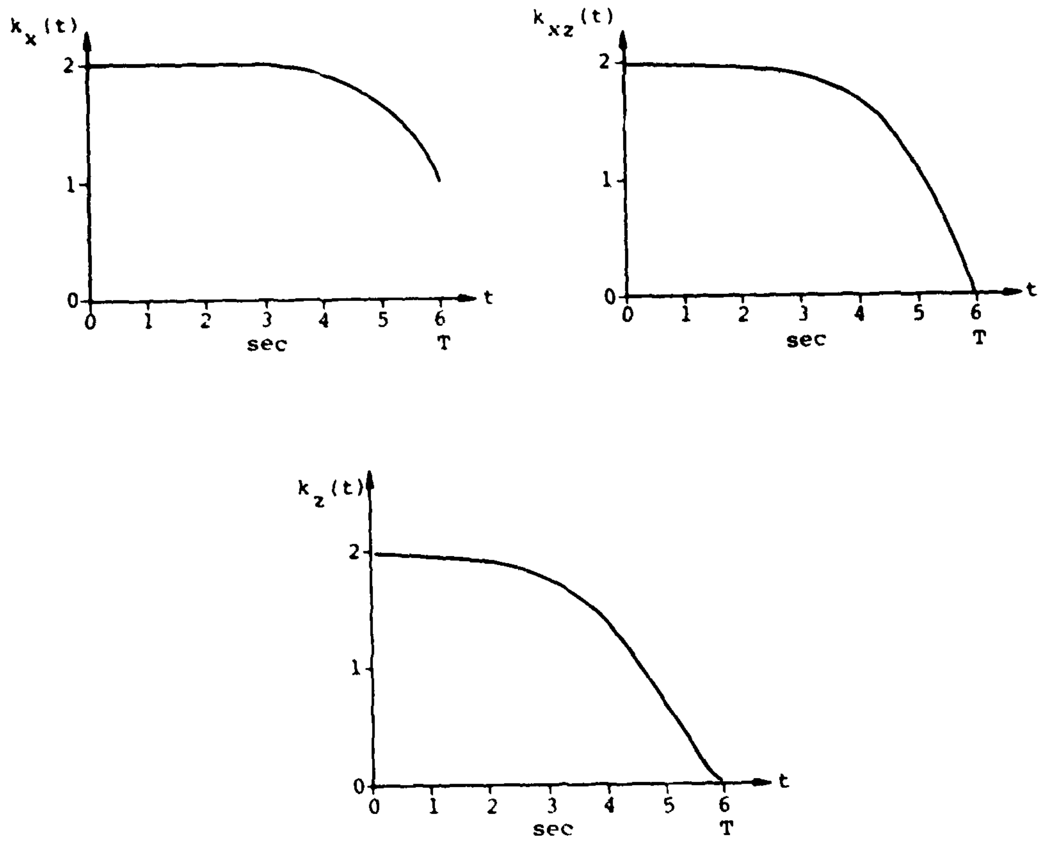

Figure 3-5. Numerical gain results for scalar zero set-point regulator with $s = 1$, $q = o$, $r = 1$, $T = 6.0$.

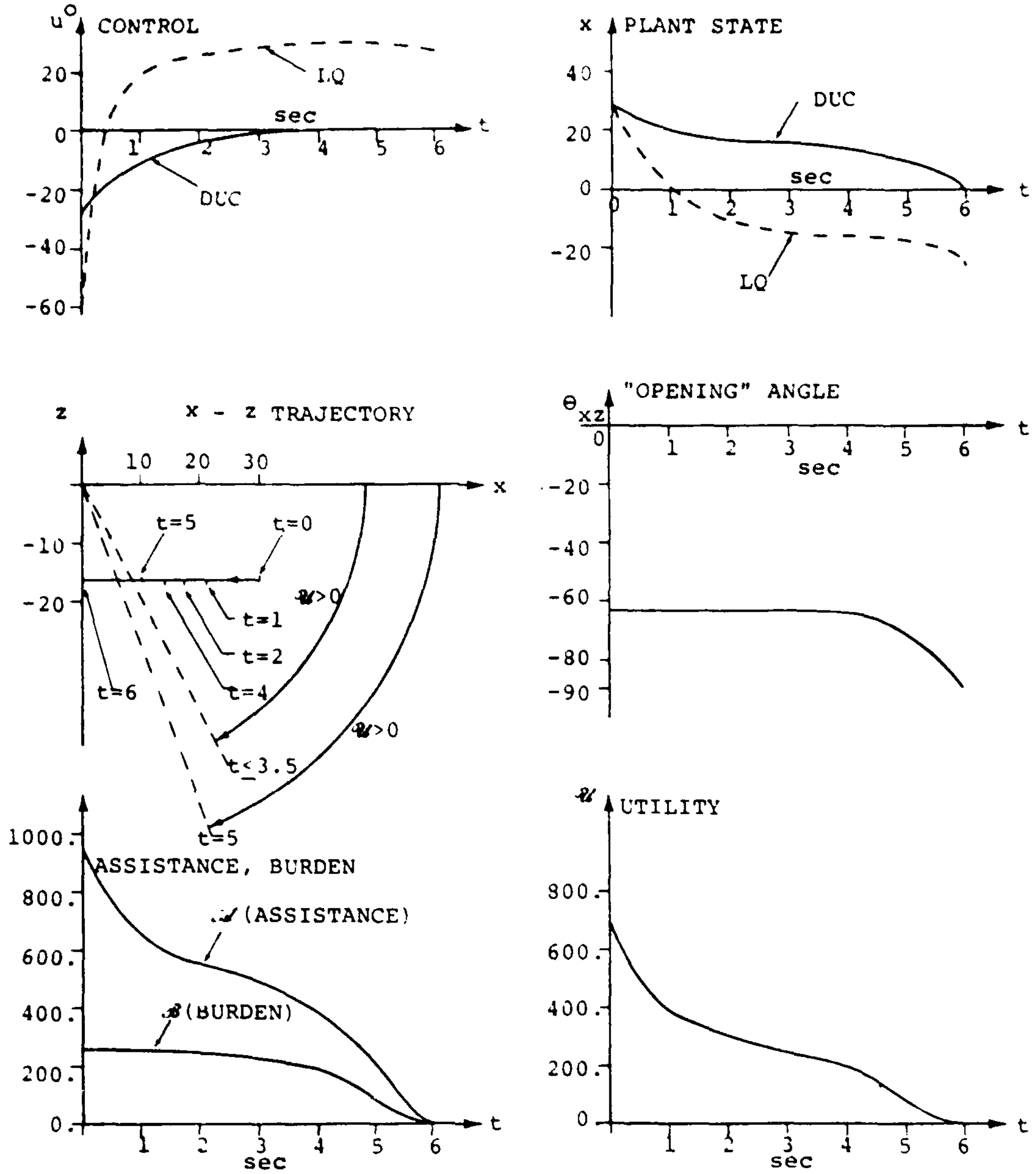

Figure 3-6. Computed performance of disturbance-utilizing scalar regulator with zero set-point, constant disturbance; s = 1, q = o, r = 1, w(t) = -16.1, x(o) = 30.0, T = 6.0.

TABLE 3-3. COMPARISON OF PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER AND CONVENTIONAL LQ CONTROLLER; SCALAR ZERO SET-POINT REGULATOR, WITH CONSTANT DISTURBANCE, $w(t) = -16.1$, $x_{sp} = 0.0$, $s = 1$, $q = o$, $r = 1$, $T = 6.0$.

	J_m	J_q	J_r	$J(T)$	δ_T %	$e = x_{sp} - x(T)$ (FT)	δ_M	EU %	δ_E %
DUC	0.0021	0.	192.4	192.4	92.3	-0.065	99.7	192.	91.1
LQ	313.6	0.	2155.	2469.		25.05		2156.	

$$J(T) = J_m + J_q + J_r$$

$$= \tfrac{1}{2} s e^2(T) + \tfrac{1}{2}\int_0^T q e^2(t)\,dt + \tfrac{1}{2}\int_0^T r\, u^{o2}(t)\,dt$$

$$u^o\big|_{DUC} = -\frac{b}{r}\left(k_x x + k_{xz} z\right)$$

$$u^o\big|_{LQ} = \frac{-b}{r} k_x x$$

$$EU \triangleq \tfrac{1}{2}\int_0^T \left[u^o(t)\right]^2 dt$$

$$\delta_T \triangleq \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

$$\delta_M \triangleq \frac{|x_{sp} - x(T)|_{LQ} - |x_{sp} - x(T)|_{DUC}}{|x_{sp} - x(T)|_{LQ}} \times 100\%$$

$$\delta_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

"Total effectiveness" $\mathscr{E}_T$ is defined as

$$\mathscr{E}_T = \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\% \quad . \tag{3.60}$$

"Miss-distance effectiveness" $\mathscr{E}_M$ is defined as

$$\mathscr{E}_M = \frac{|x_{sp} - x(T)|_{LQ} - |x_{sp} - x(T)|_{DUC}}{|x_{sp} - x\ (T)|_{LQ}} \times 100\% \tag{3.61}$$

"Energy effectiveness" $\mathscr{E}_E$ is defined as

$$\mathscr{E}_E = \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\% \tag{3.62}$$

Positive values of $\mathscr{E}_T$, $\mathscr{E}_M$ or $\mathscr{E}_E$ mean that the disturbance-utilizing controller has achieved a lower value of J, absolute miss-distance or control energy consumption, respectively, than the LQ controller. The maximum possible values of $\mathscr{E}_T$, $\mathscr{E}_M$ or $\mathscr{E}_E$ are, of course, 100%.* Note, in Table 3-3, that the DAC controller obtains very large positive values of $\mathscr{E}_T$, $\mathscr{E}_M$ and $\mathscr{E}_E$. The disturbance utilizing controller for this example uses less than 9 percent of the control energy required by the conventional LQ controller. Moreover, the disturbance-utilizing controller achieves a set-point error of -0.065 feet -- only 0.3% of the 25.05 feet set-point error of the LQ controller.

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*Note, for example, that if J_{DUC} is 10% of J_{LQ}, $\mathscr{E}_T$ = 90%; if J_{DUC} is 90% of J_{LQ}, $\mathscr{E}_T$ = 10%.

3.2.2.1.2 Positive Disturbance, Positive $x(o)$. The previous case involved a constant negative disturbance which resulted in a trajectory lying inside the positive-utility domain in the x-z plane. We now consider the case in which the sign of the disturbance is reversed; that is, where $w(t)$ has a constant positive value, with all other conditions remaining exactly the same as in the previous case. The time functions $k_x(t)$, $k_{xz}(t)$ and $k_z(t)$ are exactly the same as for the previous case (see Figure 3-5), since they do not depend on the specific waveform of the disturbance. The performance for this case is summarized in Figure 3-7 and Table 3-4. In this example, the initial conditions $(x(o), z(o))$ lie in the negative-utility domain (in the first quadrant). However, the disturbance-utilizing control forces the x-z state trajectory into the second quadrant, where positive utility is available. The disturbance-utilizing control maneuvers the system so that positive utility is "made available" for this case during the time interval following 1.8 seconds. Figure 3-7 shows how the x-z trajectory "doubles back" on itself between 1 and 6 seconds. Although the state is driven to zero at $t = 1$, the terminal-time specification is not satisfied until $t = 6$, when the problem ends. The DUC uses the "extra time" between 1 and 6 seconds to place the trajectory within the $\mathcal{U} > o$ sector of the x-z plane. Note the bias effect (Figure 3-7) of the disturbance on the LQ control, which results in a large positive value of $x(T)$.

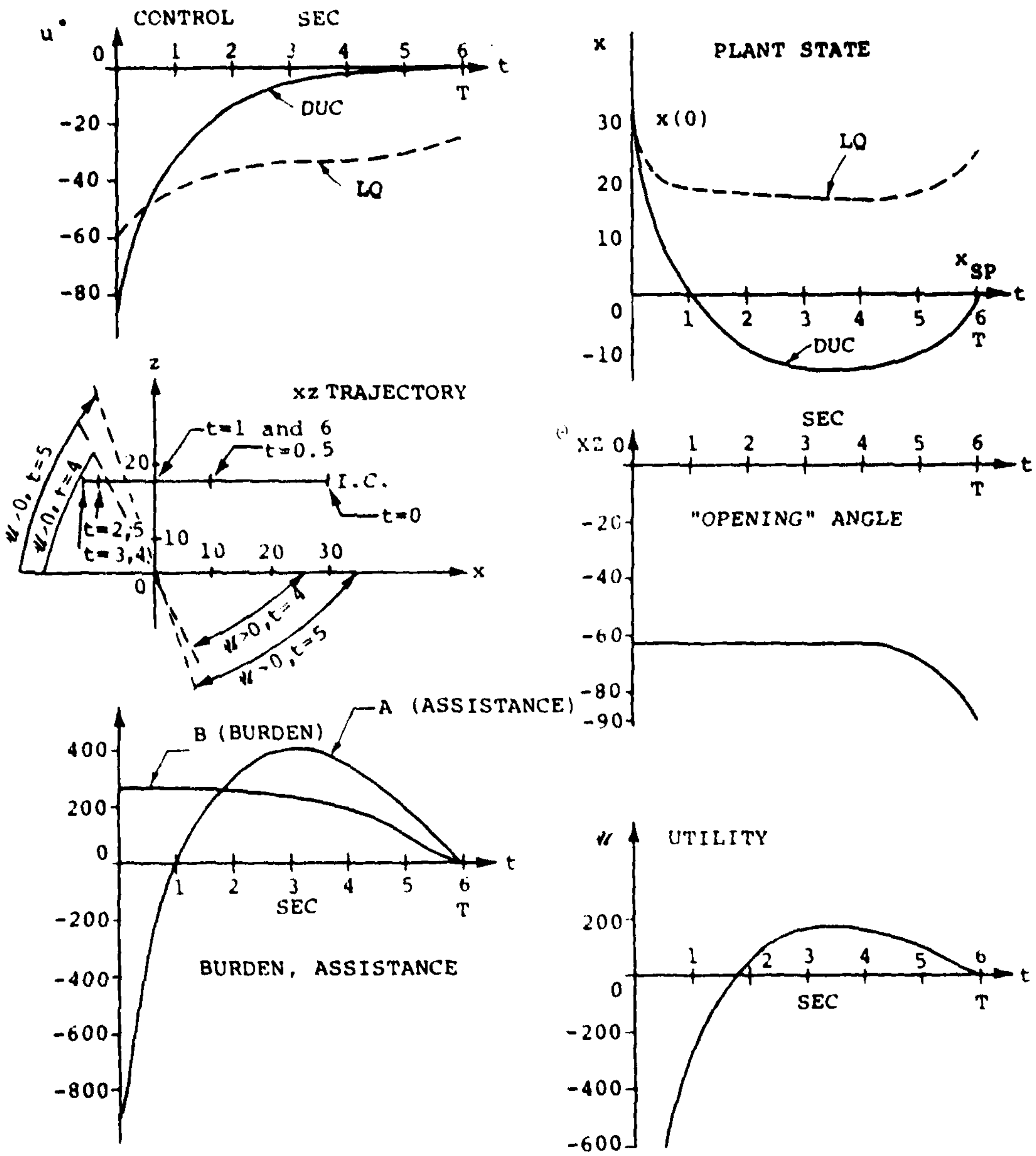


Figure 3-7. Computed performance of disturbance-utilizing scalar regulator with zero set-point, constant disturbance, s = 1, q = o, r = 1, w(t) = +16.1, x(o) = 30.0, T = 6.0.

TABLE 3-4. COMPARISON OF PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER AND CONVENTIONAL LQ CONTROLLER; SCALAR ZERO-SET-POINT REGULATOR WITH CONSTANT DISTURBANCE, $w(t) = +16.1$, $x_{sp} = 0.$, $x(o) = 30.0$, $s = 1$, $q = o$, $r = 1$, $T = 6.0$.

	J_M	J_Q	J_R	$J(T)$	$\mathcal{E}_T$ %	$x^e_{sp} - x(T)$ (ft)	$\mathcal{E}_M$ %	EU	$\mathcal{E}_E$ %
DUC	0.023	0.	2100.	2100.	52.0	-0.21	99.2	2100.	48.2
LQ	320.7	0.	4056.	4377.	$\boxtimes$	-25.33	$\boxtimes$	4056.	$\boxtimes$

$$J(T) = J_m + J_q + J_r$$

$$= \tfrac{1}{2} s e^2(T) + \tfrac{1}{2} \int_0^T q e^2(t) dt + \tfrac{1}{2} \int_0^T r\, u^{o2}(t)\, dt$$

$$u^o|_{DAC} = -\frac{b}{r}(k_x x + k_{xz} z)$$

$$u^o|_{LQ} = -\frac{b}{r} k_x x$$

$$EU \triangleq \tfrac{1}{2} \int_0^T \left[u^o(t)\right]^2 dt$$

$$\mathcal{E}_T \triangleq \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

$$\mathcal{E}_M \triangleq \frac{|x_{sp} - x(T)|_{LQ} - |x_{sp} - x(T)|_{DUC}}{|x_{sp} - x(T)|_{LQ}} \times 100\%$$

$$\mathcal{E}_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

This may be compared with the previous case, where the negative disturbance caused the LQ controller to have a large negative x(T).

The terminal-time performance for this case (Table 3-4) shows that the effectiveness parameters δ_T, δ_M and δ_E are all positive, indicating that the DUC performs better than the LQ controller on the basis of J(T), miss-distance and control energy consumption. The control energy requirements are considerably larger for both the LQ and the DUC in this case, because the amount of positive utility is relatively small. However, the disturbance-utilizing controller expends only 51.8% as much control energy as the LQ controller. Moreover, the DUC accommodates the disturbance to produce a set-point error of -0.21 feet versus -25.33 feet for the LQ controller -- a miss distance only 0.8% as large as the LQ miss distance.

3.2.2.1.3 Some Effects of Varying q/r. The importance of the ratio q/r has been seen in connection with the determination of the time-varying gains for the disturbance-utilizing controller. When q = o, varying r has very little effect on control energy EU, but when q > o, the q/r ratio affects not only EU, but also J_q, J_r and x_{sp} - x(T). Table 3-5 compares the performance for three different ratios of q/r in the disturbance-utilizing example under consideration. Except for the q and r values, the conditions of this example are identical with those of section 3.2.3.1.1, where $w(t) = -16.1$. When q is increased from q = o to q = 1, less relative weight is placed on terminal set-point error in the performance index J, with the result that the terminal miss-distance increases to 3.31 feet. Another effect is seen in the increased control energy EU required to keep the squared state $x^2(t)$ closer to zero during the control interval [o,T]. When the q/r ratio is changed to 1/10, the control energy requirement is brought back down to a value only slightly larger than for the case of q = o; but the terminal miss-distance then increases slightly because its relative weighting has decreased even further. This example illustrates the various cost trade-offs which one must consider when choosing values of s, q and r in disturbance-utilizing regulator designs.

3.2.2.2 Non-Zero Set-Point Regulator; Scalar Plant, Constant Disturbance. When the scalar system of the previous examples is operated as a non-zero set-point regulator, with a constant disturbance $w(t) = -16.1$ and a set-point $x_{sp} = -10.$, the results of Figures 3-8 and 3-9 and Table 3-6 are obtained. Note that the $k_x(t)$, $k_{xz}(t)$ and $k_z(t)$ time functions in Figure 3-8 are identical with those of the previous examples (see Figure 3-5). In addition, the $k_{xc}(t)$, $k_c(t)$ and $k_{cz}(t)$ gains are relevant to the non-zero set-point case considered here.

TABLE 3-5. THE EFFECT OF VARYING q/r ON THE DISTURBANCE-UTILIZING CONTROLLER; SCALAR ZERO SET-POINT REGULATOR WITH CONSTANT DISTURBANCE, $w(t) = -16.1$, $x_{sp} = 0.$, $x(o) = 30.0$, $T = 6.0$

s	q	r	J_m	J_q	J_r	J(T)	$x_{sp} - x(T)$	EU
1	0	1	0.0021	0.	192.4	192.4	-0.065	192.
1	1	1	5.48	359.8	364.3	729.6	3.31	364.
1	1	10	10.80	727.7	1970.	2708.	4.65	197.

$$J(T) = J_m + J_q + J_r$$

$$= \tfrac{1}{2} s e^2(T) + \tfrac{1}{2}\int_{t_o}^{T} q e^2(t)dt + \tfrac{1}{2}\int_{t_o}^{T} r\, u^{o2}(t)dt$$

$$EU = \tfrac{1}{2}\int_{o}^{T} \left[u^o(t)\right]^2 dt$$

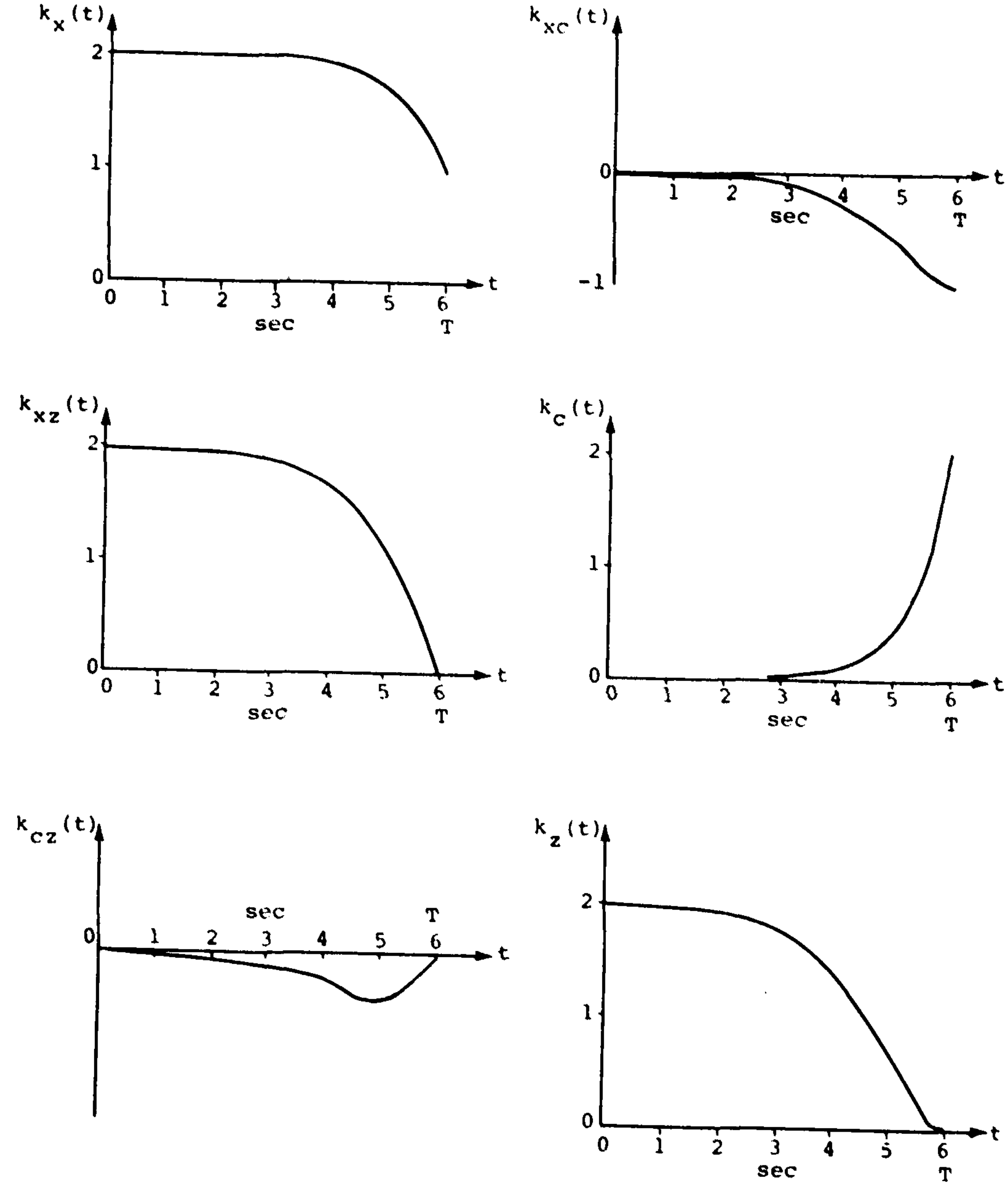


Figure 3-8. Numerical gain solutions for case of: scalar plant, scalar disturbance, non-zero set-point, constant disturbance, s = 1, q = o, r = 1, T = 6.0.

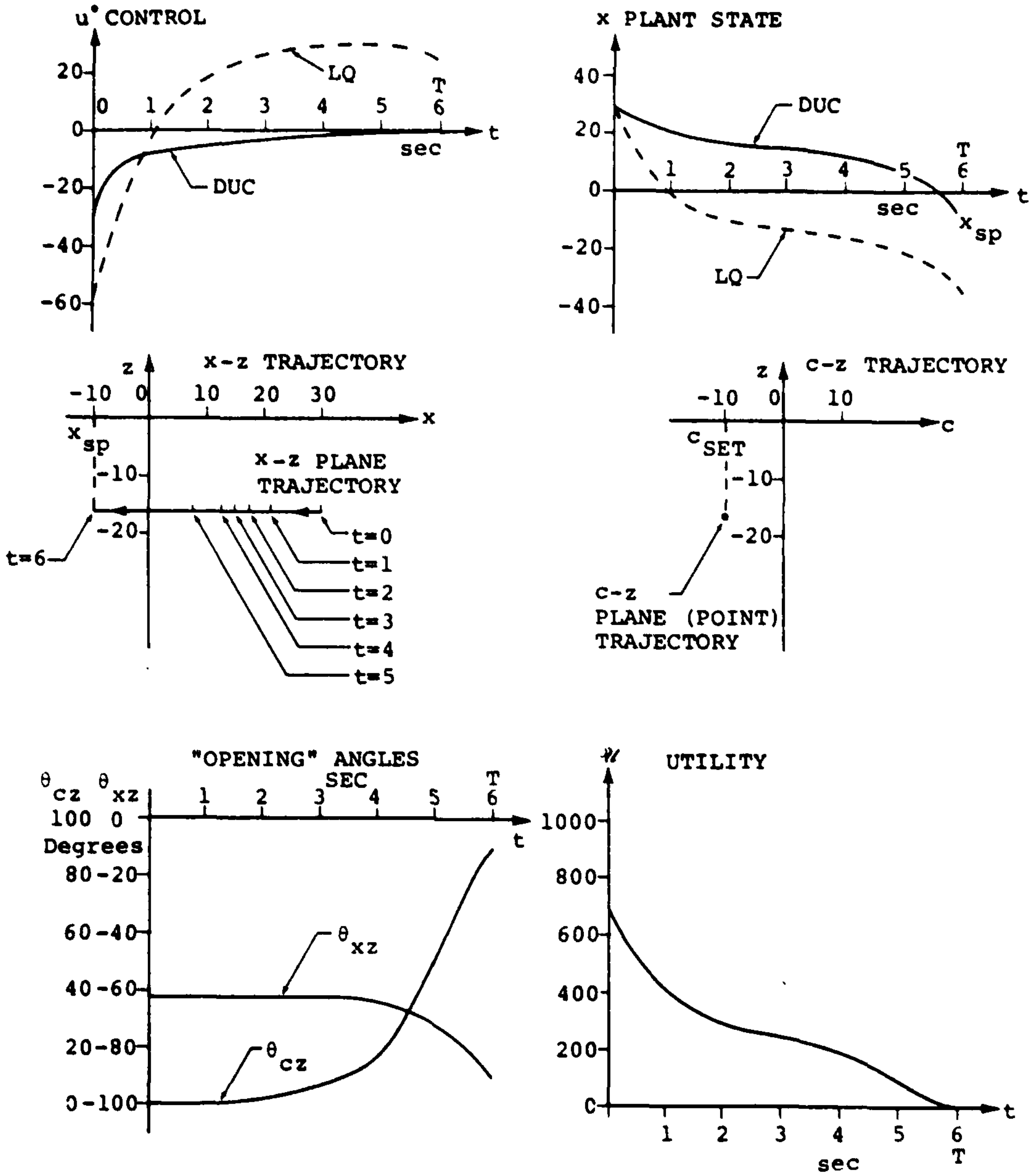


Figure 3-9. Computed performance of disturbance-utilizing scalar regulator with non-zero set-point, constant disturbance; s = 1, q = o, r = 1, w(t) = -16.1, x_{sp} = -10.0, x(o) = 30.0, T = 6.0.

TABLE 3-6. COMPARISON OF PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER AND CONVENTIONAL LQ CONTROLLER; SCALAR REGULATOR WITH NON-ZERO SET-POINT AND CONSTANT DISTURBANCE; $w(t) = -16.1$, $x_{sp} = -10.0$, $x(o) = 30.0$, $s = 1$, $q = o$, $r = 1$, $T = 6.0$

	J_m	J_q	J_r	$J(T)$	$\mathcal{E}_T$ %	$x_{sp} - x(T)$	$\mathcal{E}_M$ %	EU	$\mathcal{E}_E$ %
DUC	0.2122×10^{-2}	0.	193.1	193.1	92.2	-0.065	99.7	193.1	91.0
LQ	313.6	0.	2155.	2469.	$\boxtimes$	25.05	$\boxtimes$	2155.	$\boxtimes$

$$J(T) = J_m + J_q + J_r$$

$$= \tfrac{1}{2}\, s e^2(T) + \tfrac{1}{2} \int_{to}^{T} q e^2(t)\,dt + \tfrac{1}{2} \int_{to}^{T} r\, u^{o2}(t)\,dt$$

$$u^o|_{DAC} = -\frac{b}{r}\,(k_x x + k_{xc} c + k_{xz} z)$$

$$u^o|_{LQ} = -\frac{b}{r}\,(k_x x + k_{xc} c)$$

$$\mathcal{E}_T \triangleq \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

$$\mathcal{E}_M \triangleq \frac{|x_{sp} - x(T)|_{LQ} - |x_{sp} - x(T)|_{DUC}}{|x_{sp} - x(T)|_{LQ}} \times 100\%$$

$$\mathcal{E}_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

Figure 3-9 shows that the DUC obtains positive utility for the 6 second control interval (except at $t = T$). The x-z trajectory remains in the positive utility domain of the x-z plane for the first 5.5 seconds of the control interval as it moves through the fourth quadrant toward the set-point ($x_{sp} = -10$). The c-z "trajectory" for this example is the point ($c = -10$, $z = -16.1$). This point "trajectory" lies in the positive utility domain of the c-z plane during the last 0.9 seconds of the control interval. This may be verified by noting that θ_{cz} is greater than 58.2 degrees ($\tan^{-1}$ 16.1/10.0) for the last 0.9 seconds, which means that the c-z point "trajectory" is inside the positive utility domain. The net result of the actions of the set-point "command" and the disturbance is that positive utility is "made available" for the whole control interval. The constant disturbance has a very detrimental effect on the LQ control and state histories (Figure 3-9), resulting in a large set-point error. The terminal-time performance for this example is shown in Table 3-6. Note that the control energy EU required by the DUC in this example is only slightly larger than that required by the zero set-point regulator (compare with Table 3-3). This is apparently due to the fact (observed above) that the x-z trajectory and the c-z trajectory are complementary, in terms of the respective time intervals during which they lie in positive utility domains. The effectiveness parameters $\mathcal{E}_T$, $\mathcal{E}_M$ and $\mathcal{E}_E$, are all positive and quite large, reflecting the superior performance of the DUC in terms of performance index J, set-point error and energy consumption.

3.3 Scalar Regulator with an Exponentially-Decaying Disturbance

3.3.1 General Results. In this example we consider the scalar system of Equations (3.1) and (3.2), where the disturbance w(t) is exponentially decaying, so that

$$w(t) = C_1 e^{-\alpha t} \quad , \quad \alpha > 0 \tag{3.63}$$

$$w = Hz \quad , \quad H = [1] \tag{3.64}$$

$$\dot{z} = Dz + \sigma(t), \quad D = -\alpha \tag{3.65}$$

where $\sigma(t)$ is a sparse sequence of impulses. The dual control objectives of set-point regulation and efficient utilization of available "disturbance energy" are to be realized, as before, by minimizing the quadratic performance index J (Equation (3.5)). The set-point is represented as the output of the dynamic system Equations (3.6) and (3.7), and the control will be found as in Equation (3.16). The set of unilaterally-coupled differential Equations (2.21) - (2.26) for this example turns out to be

$$\dot{k}_x = (-1 + \frac{1}{r} k_x) k_x - k_x - q \; ; \quad k_x(T) = s \tag{3.66}$$

$$\dot{k}_{xc} = (-1 + \frac{1}{r} k_x)\ k_{xc} + q\ ;\quad k_{xc}\ (T) = -s \tag{3.67}$$

$$\dot{k}_{xz} = (-1 + \alpha + \frac{1}{r} k_x) k_{xz} - k_x\ ;\quad k_{xz}\ (T) = 0 \tag{3.68}$$

$$\dot{k}_c = \frac{1}{r} k_{xc}^{\ 2} - q\ ;\quad k_c\ (T) = s \tag{3.69}$$

$$\dot{k}_{cz} = \alpha k_{cz} + (-1 + \frac{1}{r} k_{xz})\ k_{xc}\ ;\quad k_{cz}(T) = 0 \tag{3.70}$$

$$\dot{k}_z = 2\alpha k_z + \frac{1}{r} k_{xz}^2 - 2k_{xz}\ ;\quad k_z\ (T) = 0 \tag{3.71}$$

Equations (3.66) - (3.71) correspond to the general Equations (2.21) - (2.26) with $A = 1$, $B = 1$, $R = r$, $C = 1$, $Q = q$, $S = s$, $E = o$, $G = 1$, $F = 1$, $H = 1$ and $D = -\alpha$. The corresponding equations in backward time ($\tau = T - t$) are

$$\dot{k}_x(\tau) = (\ 1 - \frac{1}{r} k_x(\tau))\ k_x(\tau) + k_x(\tau) + q;\quad k_x\ (0) = s \tag{3.72}$$

$$\dot{k}_{xc}(\tau) = (\ 1 - \frac{1}{r} k_x(\tau))\ k_{xc}(\tau) - q\ ;\ k_{xc}(0) = -s \tag{3.73}$$

$$\dot{k}_{xz}(\tau) = (1 - \alpha - \frac{1}{r} k_x(\tau))\ k_{xz}\ (\tau) + k_x\ (\tau)\ ; \quad k_{xz}\ (0) = 0 \tag{3.74}$$

$$\dot{k}_c(\tau) = -\frac{1}{r} k_{xc}^{\ 2} + q\ ;\ k_c\ (0) = s \tag{3.75}$$

$$\dot{k}_{cz}(\tau) = -\alpha k_{cz} + (1 - \frac{1}{r} k_{xz}) k_{xc} \; ; \quad k_{cz}(0) = 0 \qquad (3.76)$$

$$\dot{k}_{z}(\tau) = -2\alpha k_{z} - \frac{1}{r} k_{xz}^{2} + 2k_{xz} \; ; \; k_{z}(0) = 0 \; . \qquad (3.77)$$

As in the case of the constant disturbance, the steady-state values of $k_x(\tau)$ and $k_{xc}(\tau)$ are found to be

$$\bar{k}_x = r(1 + \sqrt{q/r + 1}) \qquad (3.78)$$

$$\bar{k}_{xc} = \frac{-q}{\sqrt{q/r + 1}} \; . \qquad (3.79)$$

Equation (3.74) has the solution

$$k_{xz}(\tau) = \frac{\bar{k}_x}{(1 - \alpha - \frac{1}{r}\bar{k}_x)} \left(-1 + e^{(1 - \alpha - \frac{1}{r}\bar{k}_x)\tau} \right) \qquad (3.80)$$

and

$$\bar{k}_{xz} = \lim_{\tau \to \infty} k_{xz}(\tau) = \frac{-\bar{k}_x}{(1 - \alpha - \frac{1}{r}\bar{k}_x)} \; . \qquad (3.81)$$

When $\bar{k}_x$ from Equation (3.78) is substituted in Equation (3.81) the result is

$$\bar{k}_{xz} = \frac{r\,(1 + \sqrt{q/r + 1})}{\alpha + \sqrt{q/r + 1}} \;. \tag{3.82}$$

As $\tau \to \infty$ it is found that no steady-state value exists for $k_c(t)$ for general values of q and r. However, the asymptotic slope $\bar{\dot{k}}_c$ does exist in backward time, and is given by

$$\bar{\dot{k}}_c = \lim_{\tau \to \infty} \dot{k}_c\,(\tau) = \frac{q^2}{q + r} \tag{3.83}$$

It is easily shown that Equation (3.76) has the solution

$$k_{cz}(\tau) = \frac{1}{\alpha}\,\bar{k}_{xc}\left(\frac{1}{r}\,\bar{k}_{xz} - 1\right)\left[-1 + e^{-\alpha\tau}\right] \tag{3.84}$$

and therefore

$$\bar{k}_{cz} = \lim_{\tau \to \infty} k_{cz}(\tau) = -\frac{1}{\alpha}\,\bar{k}_{xc}\left(\frac{1}{r}\,\bar{k}_{xz} - 1\right) \tag{3.85}$$

Substituting the known values of $\bar{k}_{xc}$ and $\bar{k}_{xz}$ into Equation (3.85) gives

$$\bar{k}_c = \frac{q/\alpha}{\sqrt{q/r + 1}} \left[\frac{1 - \alpha}{\alpha + \sqrt{q/r + 1}}\right] . \qquad (3.86)$$

When the expression for $\bar{k}_{xz}$ is substituted in Equation (3.77) the result is

$$\dot{k}_z = -2\alpha k_z - r\left[\frac{1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}}\right]^2 + 2\left[\frac{r(1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}}\right] . \qquad (3.87)$$

Solution of (3.87), subject to the terminal condition $k_z\ (\tau = o) = k_z\ (t = T) = o$, yields the expression

$$k_z(\tau) = \frac{(1-e^{-2\alpha\tau})}{2\alpha}\left\{- r\left[\frac{1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}}\right]^2 + 2r\left[\frac{1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}}\right]\right\} . \qquad (3.88)$$

In the limit $\tau \to \infty$, Equation (3.88) yields the steady-state value

$$\bar{k}_z = \lim_{\tau\to\infty} k_z(\tau) = \frac{r}{2\alpha}\left\{-\left[\frac{1+\sqrt{q/r+1}}{\alpha+\sqrt{q/r+1}}\right]^2 + 2\left[\frac{1+\sqrt{q/r+1}}{\alpha+\sqrt{q/r+1}}\right]\right\} \tag{3.89}$$

which is positive for all $q \geq o$, $r > o$, $\alpha \geq o$. Table 3-7 summarizes the steady-state values of gains for the general case and for the special case of $q = o$. Note the existence of steady-state values $\bar{k}_c$, $\bar{k}_{cz}$, and $\bar{k}_z$ for the special case $q = o$. These results may be compared with the constant disturbance case (Table 3-1) where $\alpha = o$. A noteworthy special case can be seen in Table 3-7 corresponding to the value $\alpha = 1$ which leads to

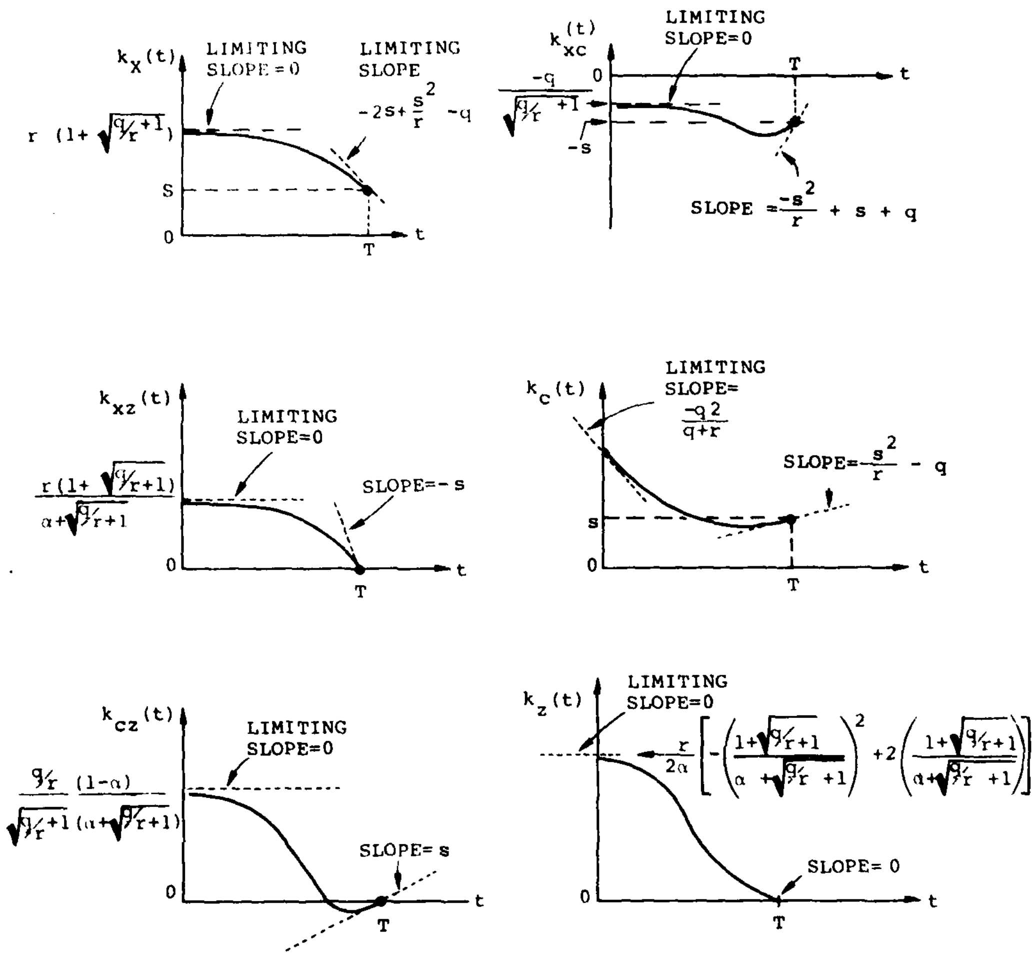
$$\bar{k}_{cz} = 0 \quad , \quad \alpha = 1 \quad , \quad q \geq 0 \quad . \tag{3.90}$$

Figure 3-10 shows typical gain histories for the case of the exponentially decaying disturbance.

For this example, steady-state values of θ_{xz} (see Table 3-8) are non-zero, for general non-negative values of q and r. This behavior for the case of an exponentially-decaying disturbance is in contrast with that for the case of the constant disturbance, where the limiting value $\bar{\theta}_{xz}$

TABLE 3-7. STEADY-STATE GAINS FOR SCALAR SET-POINT-REGULATOR WITH EXPONENTIALLY DECAYING DISTURBANCE.

	For $q \geq 0,\ r > 0,\ \alpha \geq 0$	For $q = 0,\ r > 0,\ \alpha \geq 0$
$\bar{k}_x$	$r\,(1 + \sqrt{q/r + 1})$	$2r$
$\bar{k}_{xc}$	$\dfrac{-q}{\sqrt{q/r + 1}}$	0
$\bar{k}_{xz}$	$\dfrac{r\,(1 + \sqrt{q/r + 1})}{\alpha + \sqrt{q/r + 1}}$	$\dfrac{2r}{\alpha + 1}$
$\bar{k}_c$	no limit exists	0
$\bar{k}_{cz}$	$\dfrac{q/\alpha}{\sqrt{q/r + 1}} \left(\dfrac{1 - \alpha}{\alpha + \sqrt{q/r + 1}} \right)$	0
$\bar{k}_z$	$\dfrac{r}{2\alpha} \left[-\left(\dfrac{1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}} \right)^2 + 2 \left(\dfrac{1 + \sqrt{q/r + 1}}{\alpha + \sqrt{q/r + 1}} \right) \right.$	$\dfrac{2r}{(\alpha + 1)^2}$



NOTE: "Limiting slope" means as $\tau\to\infty$; $\tau=T-t$.

Figure 3-10. Typical forward-time gain histories for a scalar set-point regulator with an exponentially-decaying disturbance and disturbance-utilizing control.

is zero for general non-negative values of q and r. The existence of finite values of $\bar{k}_z$ prevents the positive utility region from collapsing for large τ, and the larger the value of α, the larger the value of θ_{xz} as $\tau \to \infty$. For q = o, it is found that

$$\lim_{\tau \to \infty} \tan \theta_{xz}(\tau) = -2(\alpha + 1), \; q = o \tag{3.91}$$

and therefore, when α = o, the limiting value of θ_{xz} is -63.43 degrees, which is the same value obtained for the case of a constant disturbance. Thus, for the exponential disturbance, the limiting angle is more negative than -63.43 degrees.

Steady state values of θ_{cz} are particularly sensitive to α; the value α = 1 is the critical point which determines whether positive or negative values of θ_{cz} are obtained as $\tau \to \infty$*. Table 3-8 lists some typical values of $\bar{\theta}_{cz}$ and $\bar{\theta}_{xz}$.

3.3.2 A Numerical Example: A Scalar Regulator with Non-Zero Set-Point and Exponentially-Decaying Disturbance. In this example, the scalar system of the previous examples is operated as a non-zero set-point regulator with an exponentially-decaying disturbance, where

*Note that $\bar{k}_{cz}$ (in Table 3-7) changes sign as α changes from values less than 1 to values greater than 1.

TABLE 3-8. PROPERTIES OF THE POSITIVE-UTILITY DOMAIN "OPENING" ANGLES Θ_{xz} AND Θ_{cz} FOR SCALAR SET-POINT REGULATOR WITH EXPONENTIALLY DECAYING DISTURBANCE

Θ_{xz} (Degrees) for $\tau \to \infty$ (steady-state)	q	r	α
-75.96°	1	1	1
-87.44°	1	1	10
-75.96°	0	r > 0	1
-65.56°	0	1	0.1

Θ_{xz} (Degrees) for $\tau \to 0$ (at terminal time)	q	r	α
-90.00°	$q \geq 0$	$r > 0$	$\alpha \geq 0$

Θ_{cz} (Degrees) for $\tau \to \infty$ (steady-state)	q	r	α
-68.96°	1	1	0.1
+80.37°	1	1	10.00
+90.00°	$q > 0$	$r > 0$	$\alpha \to \infty$
00.00°	0	$r > 0$	$\alpha \geq 0$
00.00°	$q \geq 0$	$r > 0$	1

Θ_{cz} (Degrees) for $\tau \to 0$ (at terminal time)	q	r	α
+90.00°	$q \geq 0$	$r \geq 0$	$\alpha \geq 0$

$$w(t) = -16.1\, e^{-0.1t} \ ; \ \alpha = 0.1 \tag{3.92}$$

and $x_{sp} = -10.0$. The numerical solutions of the six unilaterally-coupled differential equations (3.66) - (3.71) for this example are shown in Figure 3-11. Note that steady-state solutions exist for all six gains in Figure 3-11. The performance of the disturbance-utilizing controller is summarized in Figure 3-12 and Table 3-9. Those results may be compared with the results obtained for the constant disturbance (Figure 3-9 and Table 3-6). The x-z trajectory for the present example lies in the positive utility domain for about the last one second. Compared with the constant disturbance case (Table 3-6) the DUC in the present case requires slightly more control energy, because less "free energy" is available in the decaying disturbance. Likewise, the LQ regulator requires less energy in the present case, when compared with Table 3-6, because it has less disturbance to overcome. The LQ controller (Figure 3-12) shows the effect of the disturbance by driving the state x to a large set-point error (although not as large as in the case of the constant disturbance). The disturbance-utilizing controller achieves large positive values of $\mathcal{E}_T$, $\mathcal{E}_M$ and $\mathcal{E}_E$, showing that the DUC is much more effective than the LQ regulator, in terms of miss-distance and energy consumption.

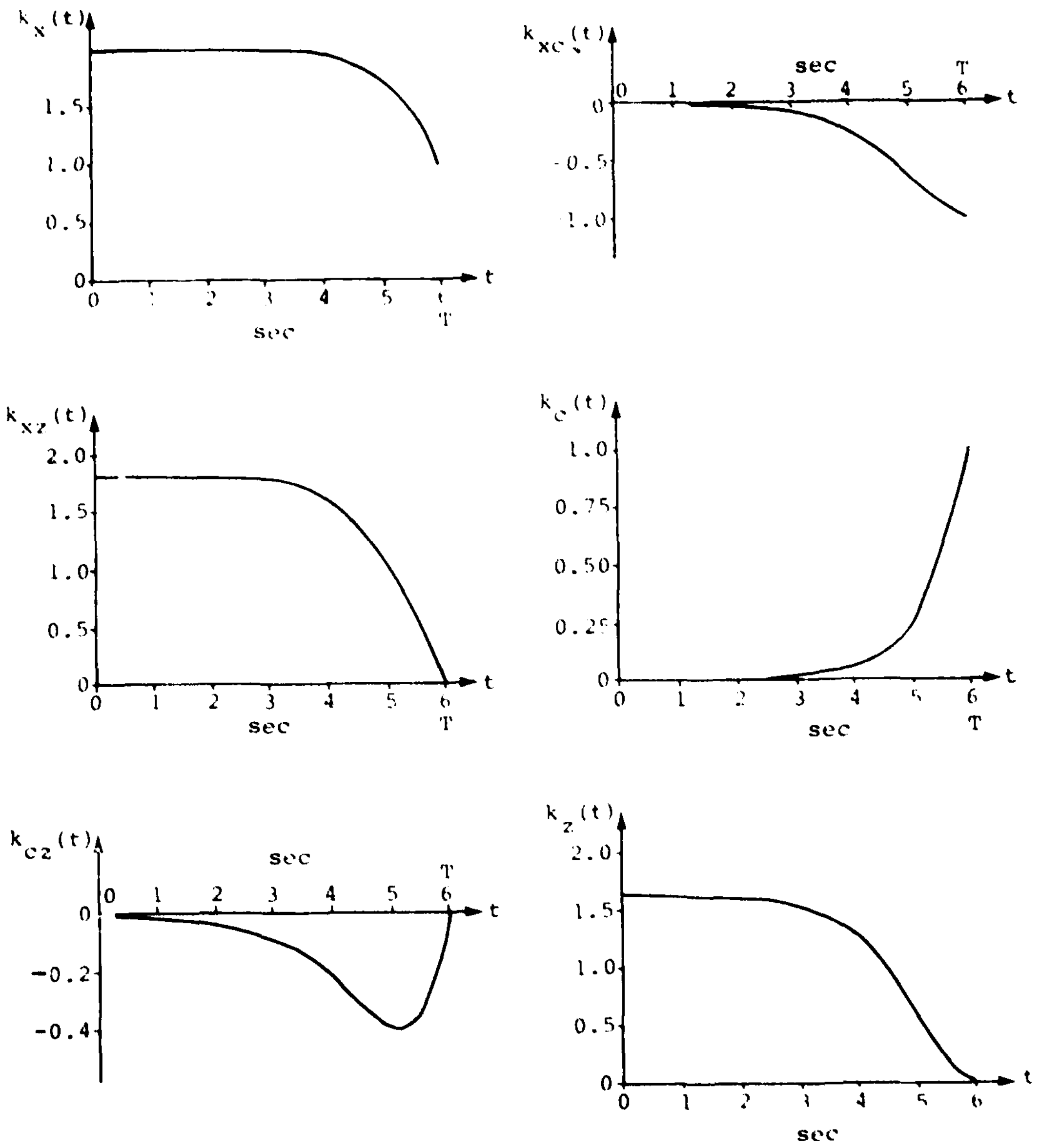


Figure 3-11. Numerical gain solutions for case of scalar set-point regulator with an exponentially-decaying disturbance and disturbance-utilizing control; $s = 1$, $q = o$, $r = 1$, $\alpha = 0.1$, $T = 6.0$.

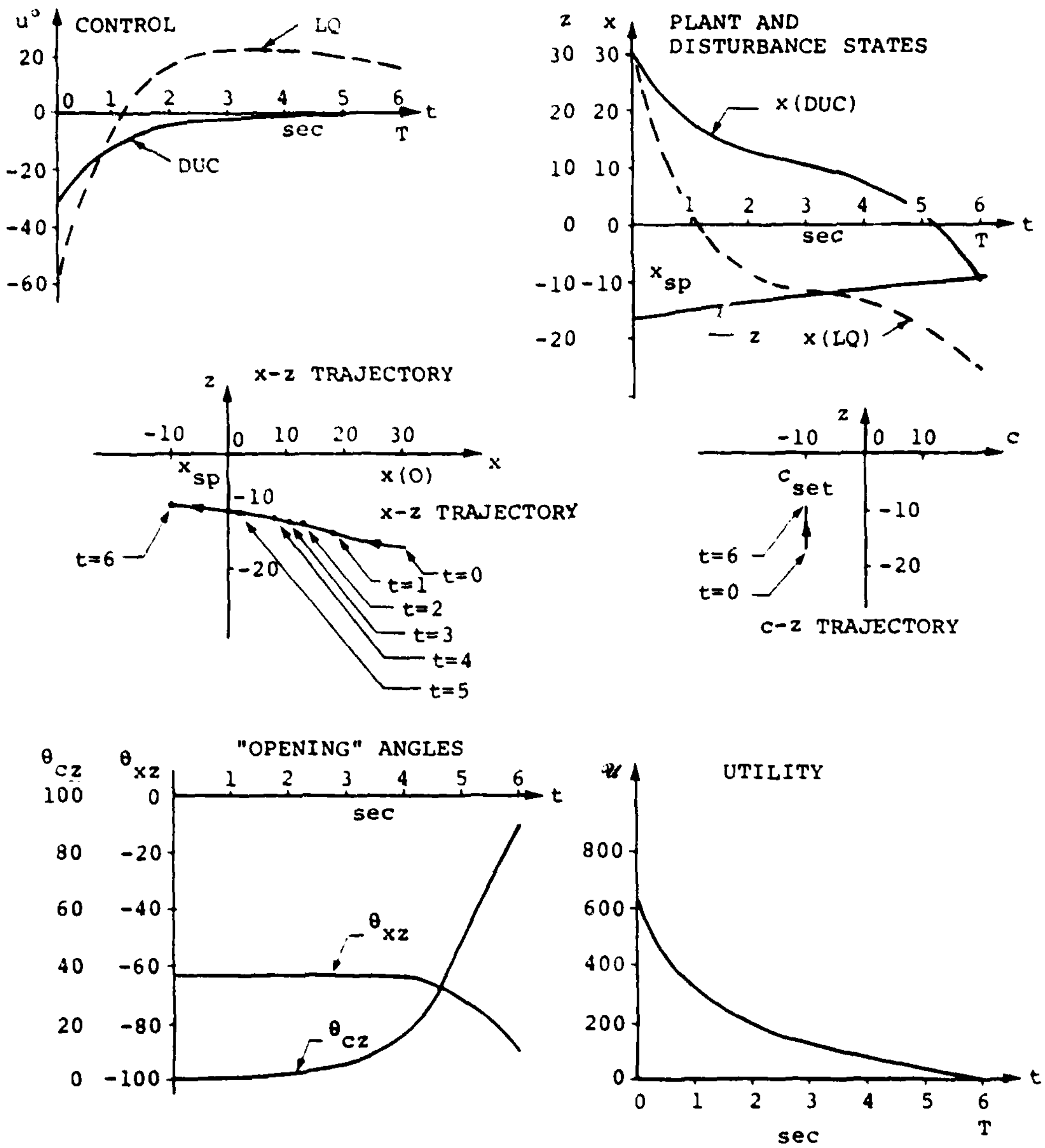


Figure 3-12. Computed performance of disturbance-utilizing scalar regulator with non-zero set-point and exponentially-decaying disturbance; s = 1, q = o, r = 1, w(t) = -16.1 exp (-0.1 t), x_{sp} = -10.0, x(o) = 30.0, T = 6.0.

TABLE 3-9. COMPARISON OF PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER AND CONVENTIONAL LQ CONTROLLER; SCALAR REGULATOR WITH NON-ZERO SET-POINT AND EXPONENTIALLY DECAYING DISTURBANCE: $w(t) = -16.1\, e^{-0.1\, t}$, $x_{sp} = -10.0$, $x(o) = 30.0$, $s = 1$, $q = o$, $r = 1$, $T = 6.0$

	J_m	J_q	J_r	$J(T)$	δ_T %	$x_{sp} - x(T)$	δ_M %	EU	δ_E %
DUC	0.2591×10^{-2}	0.0	235.0	235.0	84.2	-0.072	99.5	235.0	83.0
LQ	119.1	0.0	1373.0	1492.0		15.43		1373.0	

$$J(T) = J_m + J_q + J_r$$

$$= \tfrac{1}{2} s e^2(T) + \tfrac{1}{2} \int_{to}^{T} q e^2(t)\,dt + \tfrac{1}{2} \int_{to}^{T} r\, u^{o2}(t)\,dt$$

$$u^o_{DAC} = -\frac{b}{r}\,(k_x x + k_{xc}\, c + k_{xz}\, z)$$

$$u^o_{LQ} = -\frac{b}{r}\,(k_x\, x + k_{xc}\, c)$$

$$EU = \tfrac{1}{2} \int_{o}^{T} \left[u^o(t)\right]^2 dt$$

$$\delta_T \triangleq \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

$$\delta_M \triangleq \frac{\left|x_{sp} - x(T)\right|_{LQ} - \left|x_{sp} - x(T)\right|_{DUC}}{\left|x_{sp} - x(T)\right|_{LQ}} \times 100\%$$

$$\delta_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

3.4 A Second-Order Plant with Zero Set-Point and an Exponentially Decaying Vector Disturbance

3.4.1 General Results. In this example, we consider the system

$$\dot{x}_1 = x_2 + w_1(t) \tag{3.93}$$

$$\dot{x}_2 = u + w_2(t) \tag{3.94}$$

$$y = \left[x_1,\ x_2 \right]^T \tag{3.95}$$

which may be written in vector-matrix format as

$$\dot{x} = A\ x + B\ u + F\ w \tag{3.96}$$

$$y = C\ x \tag{3.97}$$

where the disturbance w is assumed to be a two-dimensional vector whose elements are decaying exponentials of the form

$$w_1(t) = C_1 e^{-\alpha_1 t} = z_1 \tag{3.98}$$

$$w_2(t) = C_2 e^{-\alpha_2 t} = z_2 \tag{3.99}$$

Following the disturbance-modeling procedures outlined in Chapter I, we have

$$\dot{z}_1 = -\alpha_1 z_1 + \sigma_1(t) \tag{3.100}$$

$$\dot{z}_2 = -\alpha_2 z_2 + \sigma_2(t) \tag{3.101}$$

$$w = Hz \; ; \; H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{3.102}$$

$$\dot{z} = Dz \; ; \quad D = \begin{bmatrix} -\alpha_1 & 0 \\ 0 & -\alpha_2 \end{bmatrix} \tag{3.103}$$

where σ_1, (t) and $\sigma_2(t)$ are sparsely-populated impulse sequences. The relevant system parameters for this example are

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \quad C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad F = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The two control objectives for this example are to achieve simultaneous set-point regulation of the two states $x_1(t)$, $x_2(t)$, while efficiently utilizing any available energy of the disturbances. These objectives will be achieved by minimizing the performance index

$$J = \frac{1}{2} e^T(T)Se(T) + \frac{1}{2}\int_{t_o}^{T} \left[e^T(t)Qe(t) + r\,(u^0(t))^2\right] dt \quad (3.104)$$

where $e(t) = x_{sp} - x(t)$. Q, S and r for this example are

$$Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \quad S = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \quad r = 1 .$$

As $\tau\to\infty$, steady-state values exist for the matrices $K_x(t)$, $K_{xz}(t)$ and $K_z(t)$, because the pair [A, B] is completely controllable and D is the system matrix of the asymptotically stable system Equation (3.103). Therefore, the conditions for the existence of solutions of the algebraic equations involving $\bar{K}_x$, $\bar{K}_{xz}$ and $\bar{K}_z$ are automatically satisfied, as was shown in Chapter II. The three algebraic (matric) equations to be solved for the case $\tau\to\infty$ are

$$(-A+BR^{-1}B^T\bar{K}_x) - \bar{K}_x A - C^TQC = 0 \quad (3.105)$$

$$(-A + BR^{-1}B^T\bar{K}_x)\ \bar{K}_{xz} - \bar{K}_x FH - \bar{K}_{xz}D = 0 \quad (3.106)$$

$$-(\bar{K}_zD + D^T\bar{K}_z) + \bar{K}_{xz}BR^{-1}B^T\bar{K}_{xz} - \left[(FH)^T\bar{K}_{xz} + \bar{K}_{xz}FH\right] = 0 \quad (3.107)$$

Substitution for A, B, etc. in these equations leads to twelve simultaneous algebraic equations to be solved. When these equations are solved, the following expressions are obtained for the steady-state gain matrices:

$$\bar{K}_x = \begin{bmatrix} \sqrt{2}\; r^{1/2} & r^{1/2} \\ r^{1/2} & \sqrt{2}\; r^{3/4} \end{bmatrix} \tag{3.108}$$

$$\bar{K}_{xz} = \begin{bmatrix} \dfrac{r^{1/2}(\sqrt{2}\;\alpha_1 r^{1/2} + 1)}{(\alpha_1^2 r^{1/2} + \sqrt{2}\;\alpha_1 r^{1/2} + 1)} & \dfrac{\alpha_2\, r}{(\alpha_2^2 r^{1/2} + \sqrt{2}\;\alpha_2 r^{1/2} + 1)} \\ \dfrac{r^{3/4}\,(\sqrt{2} + \alpha_1 r^{1/2})}{(\alpha_1^2\, r^{1/2} + \sqrt{2}\;\alpha_1 r^{1/2} + 1)} & \dfrac{r\,(\sqrt{2}\;\alpha_2 r^{1/2} + 1)}{(\alpha_2^2 r^{1/2} + \sqrt{2}\,\alpha_2 r^{1/2} + 1)} \end{bmatrix} \tag{3.109}$$

It is noted that for the special case $\alpha_1 = o$, $\alpha_2 = o$ Equation (3.109) yields:

$$\lim_{\substack{\alpha_1 \to 0 \\ \alpha_2 \to 0}} \bar{K}_{xz} = \begin{bmatrix} r^{1/2} & 0 \\ \sqrt{2}\; r^{3/4} & r \end{bmatrix} \tag{3.110}$$

The elements of $\bar{K}_z$ are found to be expressed as follows:

$$\bar{k}_{z_{11}} = \frac{\sqrt{2}\, r^{3/4}\left(\alpha_1^2 r^{1/2} + \dfrac{5\alpha_1}{2\sqrt{2}}\, r^{1/4} + 1\right)}{(\alpha_1^2 r^{1/2} + \sqrt{2}\,\alpha_1 r^{1/4} + 1)^2} \tag{3.111}$$

$$\bar{k}_{z_{12}} = \bar{k}_{z_{21}} =$$

$$\frac{\alpha_2 r\left[(\alpha_1\,\alpha_2 + \alpha_1^{\,2})\, r^{1/2} + \sqrt{2}\,(\alpha_1 + \alpha_2) r^{1/4} + 1\right]}{(\alpha_1 + \alpha_2)\,(\alpha_1^{\,2} r^{1/2} + \sqrt{2}\,\alpha_1 r^{1/4} + 1)\,(\alpha_2^{\,2} r^{1/2} + \sqrt{2}\,\alpha_2 r^{1/4} + 1)} \tag{3.112}$$

$$\bar{k}_{z_{22}} = \frac{r(\sqrt{2}\,\alpha_2 r^{1/4} + 1)(2\,\alpha_2^{\,2} r^{1/2} + \sqrt{2}\,\alpha_2 r^{1/4} + 1)}{2\,\alpha_2(\alpha_2^{\,2} r^{1/2} + \sqrt{2}\,\alpha_2 r^{1/4} + 1)^2} \tag{3.113}$$

3.4.2 <u>A Numerical Example</u>. Let $T = 6.0$, $C_1 = -16.1$, $C_2 = -16.1$, $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$. The latter values imply that

$$D = \begin{bmatrix} -0.1 & 0 \\ 0 & -0.7 \end{bmatrix} . \tag{3.114}$$

The computed gain matrices $K_x(t)$, $K_{xz}(t)$ and $K_z(t)$ for this example are shown in Figures 3-13 and 3-14. Note that the elements of each matrix approach steady-state values for $t \to \infty$ in this example. Of particular interest in Figure 3-13 is the fact that $k_{x_{11}}(T) = 10$, which results from the weighting of 10 on the position state in the terminal weighting matrix S. The remaining elements of K_x are zero at t = T, of course, because the remaining elements of S are zero. The DUC uses both matrices, $K_x(t)$ and $K_{xz}(t)$, while the LQ controller uses only the $K_x(t)$ matrix. The matrix $K_z(t)$ in Figure 3-14 is not required for implementing the DUC control law, but is used in computing burden and utility for analysis purposes.

The control histories for the DUC and LQ controllers are shown in Figure 3-15. The DUC accomplishes most of its control action early in the control interval, while the LQ controller must develop a significant amount of control near the end. This appears to be due to the fact that the LQ controller takes cognizance of the disturbance only through the effects that the disturbance has on the states of the plant, while the DUC has current information on the disturbance.

The plant state history (Figure 3-15) shows the action of the LQ controller in driving $x_1(T)$ to a negative value (-6.6 feet) in response to the negative disturbance; the DUC, however, drives $x_1(T)$ much closer (to within 0.05 feet) to the desired set point in the face of the same disturbance. The DUC also produces a smaller value of $x_2(T)$, although no terminal weighting was placed on $x_2(T)$.

The utility (Figure 3-15) has a large positive value at t = o, but is quickly driven negative, evidently because of

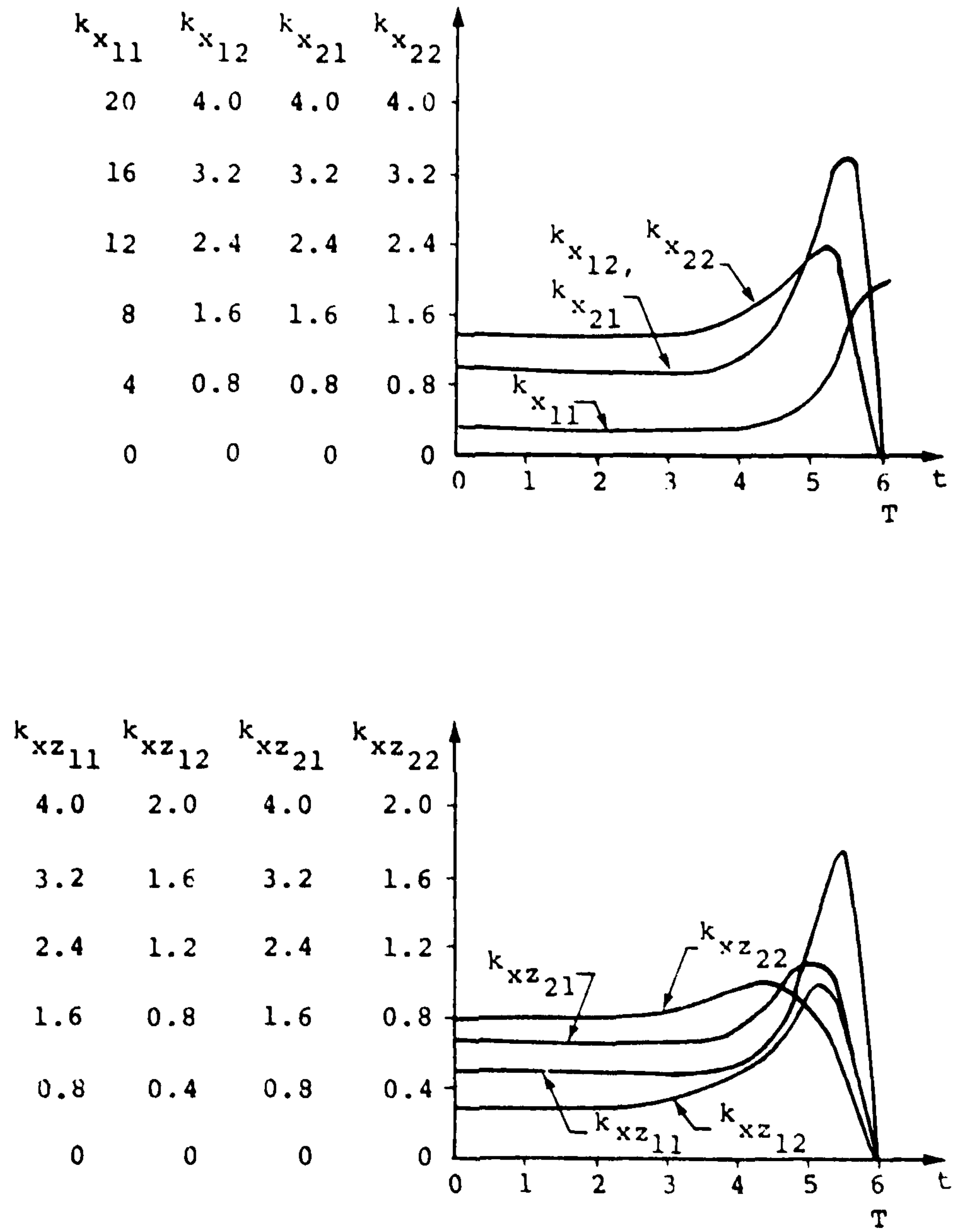


Figure 3-13. Computed gain matrices $K_x(t)$ and $K_{xz}(t)$ for a second-order plant with disturbance-utilizing controller; exponentially-decaying vector disturbance with $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$.

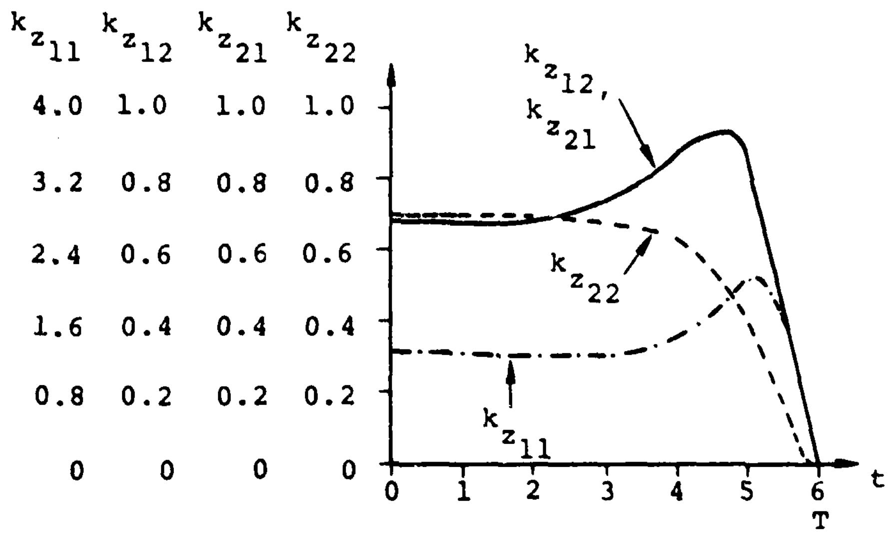


Figure 3-14. Computed gain matrix $K_z(t)$ for a second-order plant with disturbance-utilizing controller; exponentially-decaying vector disturbance with $\alpha_1 = 0.1$ and $\alpha_2 = 0.7$.

the negative value of x_2 near t = 1. After x_2 becomes positive at t = 2, the utility becomes positive again, and remains so until the terminal time at t = 6. It should be noted that this example is not designed to achieve ultimate utility, since Q is not zero. The particular Q matrix chosen in this example encourages the position state $x_1(t)$ to be driven rather quickly to a small value, as Figure 3-15 shows. There is, therefore, a trade-off between keeping $x_1(t)$ small and efficiently utilizing the disturbance in this example. Even so, a significant amount of positive utility is "made available" by the disturbance-utilizing control.

Table 3-10 shows that the DUC is more effective than the LQ controller on the basis of performance index J, position (x_1) set-point error, velocity (x_2) set-point error and control energy consumption. This is reflected in the large positive values for the effectiveness parameters δ_T, δ_{M1}, δ_{M2} and δ_E. Relative to the stated control objectives for this example, the disturbance-utilizing controller achieves a position set-point error of -0.05 feet, compared with 6.6 feet for the LQ controller and consumes only 40% as much control energy as the LQ controller.

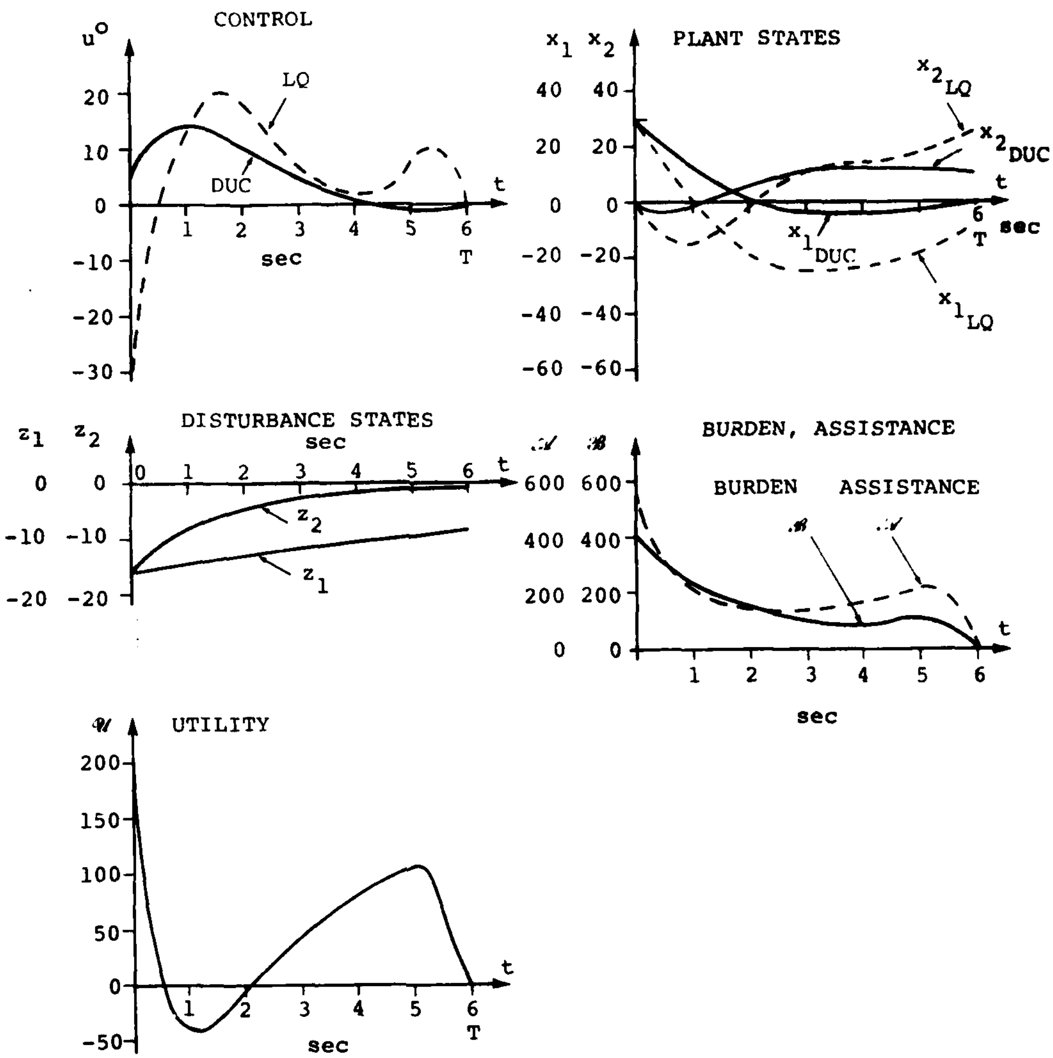


Figure 3-15. Computed performance of second-order disturbance-utilizing zero set-point regulator with asymptotically-decaying vector disturbance; $w_1(t) = -16.1 \exp(-0.1\ t)$, $w_2(t) = -16.1 \exp(-0.7\ t)$, $x_1(o) = 30.$, $T = 6.0$, $r = 1$,

$$S = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

TABLE 3-10. COMPARISON OF PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER AND CONVENTIONAL LQ CONTROLLER; SECOND-ORDER REGULATOR WITH ZERO SET-POINT, VECTOR DISTURBANCE WITH $w_1(t) = -16.1 \exp(-0.1\,t)$, $w_2(t) = -16.1 \exp(-0.7\,t)$, $x_1(o) = 30.$, $x_2(o) = 0.$, $T = 6.0$, $r = 1$,

$$S = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}, \qquad Q = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}.$$

	J_M	J_Q	J_R	$J(T)$	δ_T %	$x_{1_{SP}}-x_1(T)$ (FT)	δ_{M1} %	$x_{2_{SP}}-x_2(T)$ (F/S)	δ_{M2} %	EU	δ_E %
DUC	0.01	0.301 X10^6	174.	0.301 X10^6	55.1	-0.05	99.2	-10.5	59.5	174.	60.4
LQ	216.	0.670 X10^6	439.	0.671 X10^6	X	+6.6	X	-25.9	X	439.	X

$$J(T) = J_M + J_Q + J_R$$

$$= \tfrac{1}{2} e^T(T) S e(T) + \tfrac{1}{2}\int_0^T e^T(t) Q e(t)\,dt + \tfrac{1}{2}\int_0^T r\left[u^o(t)\right]^2 dt$$

$$u^o\big|_{DUC} = -\frac{1}{r} B^T \left[K_x x + K_{xz} z\right]$$

$$u^o\big|_{LQ} = -\frac{1}{r} B^T K_x x$$

$$\delta_T \triangleq \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\%$$

$$\delta_{M_1} \triangleq \frac{|x_{1_{SP}} - x_1(T)|_{LQ} - |x_{1_{SP}} - x_1(T)|_{DUC}}{|x_{1_{SP}} - x_1(T)|_{LQ}} \times 100\%$$

$$\delta_{M_2} \triangleq \frac{|x_{2_{SP}} - x_2(T)|_{LQ} - |x_{2_{SP}} - x_2(T)|_{DUC}}{|x_{2_{SP}} - x_2(T)|} \times 100\%$$

$$\delta_E \triangleq \frac{EU_{LQ} - EU_{DUC}}{EU_{LQ}} \times 100\%$$

CHAPTER IV

APPLICATION OF DISTURBANCE-UTILIZING CONTROL THEORY TO SOME MISSILE - INTERCEPT PROBLEMS

4.1 Summary of Chapter IV

This chapter discusses the application of disturbance-utilizing control theory to missile guidance problems consisting of interception of air defense targets and homing on ground-based targets, in the face of realistic disturbances.

The air defense problems are studied by using a general planar model and solving for the optimal (disturbance-utilizing) control forces along the interceptor missile's longitudinal and lateral axes. The problems of homing on ground-based targets are studied via a so-called "small line-of-sight angle" model and solving for the optimal (disturbance-utilizing) control forces normal to a reference line-of-sight line passing through the target position.

In each case the closed-loop performance of the miss le with disturbance-utilizing control is analyzed in terms of a performance index J and related key parameters such as terminal miss distance, control energy requirements, and fuel expenditure. In addition, the "effectiveness" of the disturbance-utilizing controller is determined, in each intercept/homing problem, by comparing its performance with that of the corresponding conventional linear-quadratic controller, under identical disturbance input conditions.

4.2 A Planar - Motion Missile - Intercept Problem with Maneuvering Target and Disturbance-Utilizing Control

4.2.1 Mathematical Model. In this section we will be concerned with problems of missile-intercept - tasks of maneuvering the position of an interceptor missile so as to coincide with the position of an air target - in the face of disturbance effects which may, or may not, be detrimental to the intercept objective. The geometry for a planar-motion version of this problem is shown in Figure 4-1, where the origin of the coordinate system is located at an arbitrarily specified ground point and where, at each time t, the target position (XT1, XT3) and missile position (XM1, XM3) are dependent on the initial conditions at time t_o and on the respective applied forces, including disturbance forces.

The forces applied to the missiles are assumed to be the net horizontal and vertical components mu_1 and mu_2, respectively, of the control force (thrust components), and the net horizontal and vertical components mw_{m1} and mw_{m2}, respectively, of the disturbance forces that may be acting on the missile. The parameter m is the mass of the missile; u_1 and u_2 are the horizontal and vertical missile accelerations, respectively, due to control forces;

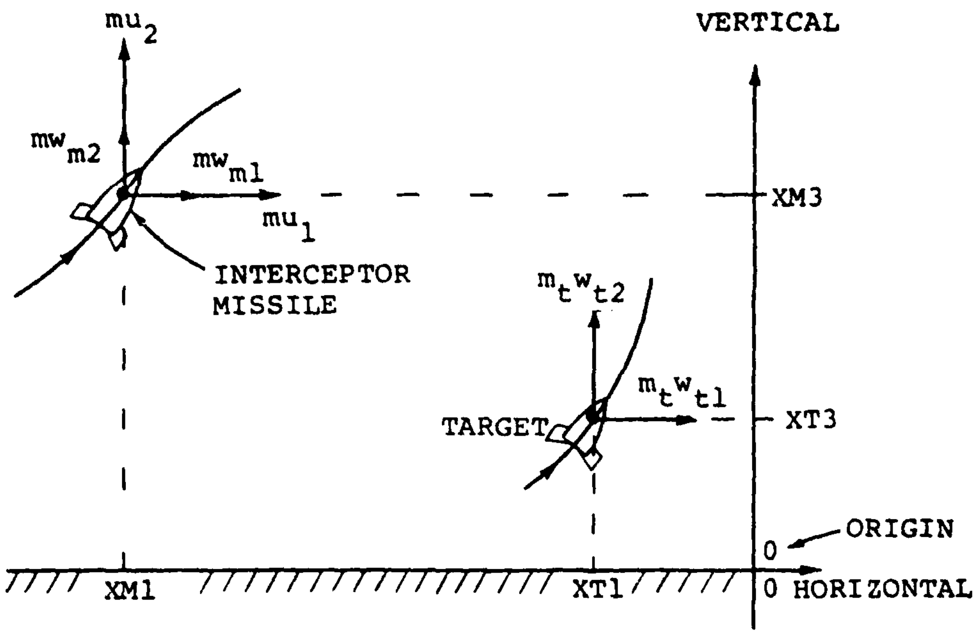


Figure 4-1. Geometry of planar-motion missile-intercept problem.

and w_{m1} and w_{m2} are the horizontal and vertical missile accelerations, respectively, due to disturbance forces.

The horizontal acceleration $w_{m1}(t)$ of the missile due to disturbances will be modeled as the superposition of two terms

$$w_{m1}(t) = \text{WIND1} + \text{D 1} \tag{4.1}$$

where WIND1 is the horizontal missile acceleration caused by wind forces and D1 is the horizontal acceleration caused by aerodynamic drag forces. Likewise, the vertical acceleration $w_{m2}(t)$ of the missile, due to disturbances, is modeled as

$$w_{m2}(t) = WIND2 + D\,2 + GRAV \tag{4.2}$$

where WIND2 is the vertical missile acceleration caused by wind forces, D2 is the vertical acceleration due to aerodynamic drag and GRAV is the acceleration of the missile due to gravity force.

It is assumed that, from the viewpoint of the interceptor, the motion of the target is "uncontrollable" and is not known a priori. Thus, the target motion may be viewed as being caused by the horizontal and vertical target disturbance forces $m_t\, w_{t1}(t)$ and $m_t\, w_{t2}(t)$, respectively.

It is convenient to formulate this intercept problem using the relative motion model

$$\dot{x}_1 = x_2 \tag{4.3}$$

$$\dot{x}_2 = u_1 + w_1(t) \tag{4.4}$$

$$\dot{x}_3 = x_4 \tag{4.5}$$

$$\dot{x}_4 = u_2 + w_2(t) \tag{4.6}$$

$$y = (x_1, x_2, x_3, x_4)^T \tag{4.7}$$

where x_1 and x_3 are the horizontal and vertical positions, respectively, of the missile relative to the target and x_2 and x_4 are the horizontal and vertical time derivatives of x_1 and x_3, respectively. The corresponding

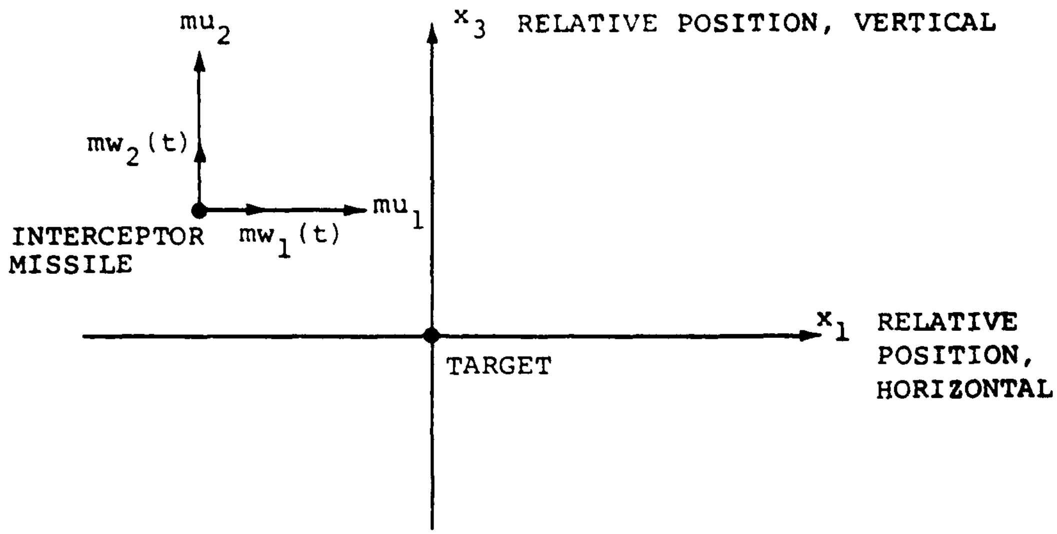


Figure 4-2. Relative motion coordinate system for planar-motion missile-intercept problem.

relative coordinate system is shown in Figure 4-2, where the origin is located at the target position.
This choice of relative coordinate system facilitates the representation of all forces acting on the target and missile as <u>net</u> forces acting solely on the missile.

The horizontal acceleration of the missile produced by this net force is

$$u_1 + w_1(t) \tag{4.8}$$

where $w_1(t)$ is the relative horizontal disturbance-induced acceleration of the missile with respect to the target, and is defined by

$$w_1(t) = w_{m1}(t) - w_{t1}(t) \tag{4.9}$$

The vertical acceleration produced by the net force on the missile is

$$u_2 + w_2(t) \tag{4.10}$$

where the disturbance-induced relative vertical acceleration of the missile with respect to the target is

$$w_2(t) = w_{m2}(t) - w_{t2}(t) \tag{4.11}$$

It is assumed that on-line, real-time measurements of each of the plant states are available from high-quality track data, so that the output vector y has x_1, x_2, x_3, and x_4 as its four elements.

Equations (4.3) - (4.7) may be written in the compact form

$$\dot{x} = A\,x + B\,u + F\,w \tag{4.12}$$

$$y = C\,x \tag{4.13}$$

where

$$A = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{4.14}$$

$$B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.15}$$

$$u = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} \tag{4.16}$$

$$F = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.17}$$

$$w = \begin{pmatrix} w_1 \\ w_2 \end{pmatrix} \tag{4.18}$$

and

$$C = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{4.19}$$

It is assumed for this problem that the unknown disturbances w_1 and w_2 can be accurately modeled as random-like linear combinations of constants and linear functions of time (ramps) as follows:

$$w_1(t) = C_1 + C_2 t \tag{4.20}$$

$$w_2(t) = C_3 + C_4 t \tag{4.21}$$

where C_1, C_2, C_3 and C_4 are unknown "constants" which may change from time-to-time. Equations (4.20) and (4.21) may thus represent, at each time t, any possible combination of bias and ramp disturbances in the control interval $[t_o, T.]$ In accordance with standard DAC protocol, it is assumed that the jumps in the C_i are not spaced "too closely" along the time axis.

The dynamic model representing the disturbance process is constructed by finding a system of differential equations which $w_1(t)$ and $w_2(t)$ satisfy. For the assumed waveform descriptions, Equations (4.20) and (4.21), such a system is

$$\dot{z}_1 = z_2 + \sigma_{11}(t) \tag{4.22}$$

$$\dot{z}_2 = \sigma_{12}(t) \tag{4.23}$$

$$\dot{z}_3 = z_4 + \sigma_{21}(t) \tag{4.24}$$

$$\dot{z}_4 = \sigma_{22}(t) \tag{4.25}$$

$$w_1 = z_1 \tag{4.26}$$

$$w_2 = z_2 \tag{4.27}$$

where the $\alpha_{ij}(t)$ represent sparsely-populated impulse sequences.

Equations (4.22) - (4.27) may be written in compact form as

$$\dot{z} = D\,z + \sigma(t) \tag{4.28}$$

$$w = H\,z \tag{4.29}$$

where

$$D = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}, \tag{4.30}$$

$$H = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} \tag{4.31}$$

and

$$\sigma(t) = \left[\sigma_{11}, \sigma_{12}, \sigma_{21}, \sigma_{22}\right]^T \tag{4.32}$$

It should be noted that there is a distinction between the assumed disturbance <u>model</u> represented by equations (4.28) and (4.29) and the disturbance <u>inputs</u> that will actually be used in later simulation runs made for this problem. The actual disturbance inputs in this problem fall into two categories:

(a) Inputs with waveform structures that are modeled exactly by Equations (4.28) and (4.29); for instance: gravity effects (a constant acceleration), and target maneuvers (a combination of ramps and constant levels of acceleration).

(b) Inputs having waveform structures that are modeled approximately by Equations (4.28) and (4.29); for instance: acceleration disturbances due to aerodynamic drag on the missile (a slowly varying nonlinear function of time), and acceleration due to wind (also a nonlinear function of time, but often a more rapidly-varying function than drag).

The specific disturbance waveforms used in the simulation studies of particular cases are discussed in Section 4.2.3.

4.2.2 Control Objectives. The planar missile-intercept problem will be formulated as a zero set-point regulator problem, where the primary control objective is to regulate the relative position states x_1 and x_3 close to zero. The secondary objective is to accomplish the primary objective in such a way as to utilize any "free" energy which may be available in disturbances such as winds, aerodynamic drag, gravity and target maneuvers.

These dual objectives will be met by minimizing the quadratic performance index

$$J = \frac{1}{2}\, e^T(T)\,Se(T) + \frac{1}{2}\int_{t_o}^{T} \left[e^T(t)\; Qe(t) + u^T(t)\,Ru(t) \right] dt \qquad (4.33)$$

where $e = x_{sp} - x = -x$, for this zero set-point case. The weighting matrices S, Q, and R used in this problem are

$$S = \begin{bmatrix} s_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & s_{33} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.34)$$

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.35)$$

$$R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \qquad (4.36)$$

where the constant values s_{11} and s_{33} are specified in the particular case considered. Recall (Section 2.3) that it is required, in general, that S and Q be symmetric, non-negative definite matrices; and it is convenient that S + Q be positive definite. However, since the plant Equations (4.3) - (4.6) for this problem are in phase variable canonical form, it is permissible to relax these conditions to allow an S matrix which has zero weighting on the "velocity" states and allow a zero Q matrix. The control objectives are achieved by minimizing J with respect to the control u(t), subject to the plant Equations (4.3) - (4.7) and the disturbance Equations (4.28) and (4.29).

4.2.3 Discussion of Results. The planar missile-intercept problem defined in Sections 4.2.1 and 4.2.2 is solved by applying the theory of Section 2.4, which leads to the composite state vector

$$\tilde{x} = \left(\frac{x}{z} \right) , \tag{4.37}$$

the composite system equation

$$\dot{\tilde{x}} = \left[\begin{array}{c|c} A & FH \\ \hline 0 & D \end{array} \right] \tilde{x} + \left[\frac{B}{0} \right] u + \left[\frac{0}{\sigma} \right] . \tag{4.38}$$

The corresponding performance index (equivalent to Equation (4.33)) is then expressed as

$$J = \frac{1}{2}\tilde{x}^T(T)\bar{S}\,\tilde{x}\,(T) + \frac{1}{2}\int_{t_o}^{T}\left[\tilde{x}^T(t)\;\bar{Q}\;\tilde{x}(t) + u^T(t)\;Ru(t)\right]dt \tag{4.39}$$

where $\bar{S} = \bar{C}^T\,S\,\bar{C}$, $\bar{C} = [-C|0]$ and $\bar{Q} = \bar{C}^T\,Q\,\bar{C}$.

The optimal control for the problem at hand is found from Equation (2.27) to be

$$u^o = -\,R^{-1}B^T\left[K_x\,x + K_{xz}\,z\right] \tag{4.40}$$

where it is assumed that the plant state x is obtained from direct measurements of the position and velocity of the missile relative to the target (as from high-quality radar tracking measurements, for example), and the disturbance state z is obtained from an ideal state reconstructor (estimator).

The assumption of an ideal disturbance state estimator for this problem may be justified on two grounds:

(a) Estimation of the disturbance state vector z may be performed with a high degree of accuracy if the estimator is designed to properly account for the class of disturbance waveforms to be encountered, assuming measurement noise is negligible. For a design that meets this criterion, the required degree of accuracy is limited primarily by the number of integrators and amplifiers used in the design, which can be implemented in large numbers by inexpensive, compact integrated circuit assemblies.

(b) The purpose of the present work is to investigate the role of disturbance utilization in missile-intercept problems. The assumption of ideal disturbance state estimates allows the analysis to be clearly focused on the disturbance-utilization aspects of the problem, as they are influenced by the waveforms of gravity, and target maneuver, without regard for effects of sensor imperfections.

The gain matrices $K_x(t)$ and $K_{xz}(t)$ in Equation (4.40) are obtained for this problem by solving the matric differential equations

$$\dot{K}_x = (-A+BR^{-1}B^TK_x)^TK_x-K_xA-C^TQC \; ; \; K_x(t)=C^TSC \qquad (4.41)$$

$$\dot{K}_{xz} = (-A+BR^{-1}B^TK_x)^TK_{xz}-K_xFH-K_{xz}D \; ; \; K_{xz}(t)=0 \; . \qquad (4.42)$$

Although not required for implementation of the control law Equation (4.40), the equation for $K_z(t)$

$$\dot{K}_z = -(K_z D + D^T K_z) + K_{xz}^T B R^{-1} B^T K_{xz} - \left[(FH)^T K_{xz} + K_{xz}^T FH\right] ; \qquad (4.43)$$

$$K_z(T) = 0$$

will also be solved to allow computation of the utility function $\mathscr{W}$ for analysis purposes. For the simulation studies of this problem, the matric functions of time $K_x(t)$, $K_{xz}(t)$ and $K_z(t)$ are obtained by forward-time solution of Equations (4.41) - (4.43) on a digital computer as t progresses from $t_o(=0)$ to T. For this purpose, a fourth-order Runge-Kutta integration routine is used on a CDC-6600 computer (integration of the plant Equation (4.12) is also performed in this way). Initial conditions for the forward integrations of Equations (4.41) - (4.43) are obtained by first performing backward-time integrations of these equations, starting at a specified terminal time T, with the known terminal conditions, and integrating back to the starting time $t_o = 0$. The element values of the respective gain matrices (K_x, K_{xz} and K_z), at $t = 0$, are then saved, to be used as the initial conditions for the subsequent forward-time runs. This procedure avoids the large computer storage requirements that would otherwise be required to store the time functions $K_x(t)$, $K_{xz}(t)$ and $K_z(t)$. A description of the digital program used for these calculations is contained in Appendix B.

The disturbance inputs actually used in the simulation studies of this problem are characterized as follows:

(a) The effect of gravity on the missile is modeled as a constant "downward" acceleration of the missile.

(b) The effect of drag on the missile is modeled as a deceleration caused by "base drag", neglecting any additional drag terms in induced by a non-zero angle of attack. The drag disturbance deceleration is simulated by means of the standard expression (directed opposite to direction of missile velocity V_m)

$$w_D = \frac{1}{2} \rho V_m^{\,2} \, S_m \, C_D/m \qquad (4.44)$$

where ρ is the air density, V_m is the total velocity of the missile

$$Vm = \sqrt{x_2^{\,2} + x_4^{\,2}} \quad ,$$

S_m is the reference area of the missile, m is the mass of the missile, obtained by a table look-up as a function of propellant burn-time (Figure 4-3), and C_D is the missile base-drag coefficient, which is obtained by a table look-up as a function of the missile mach number (Figure 4-4). The horizontal and vertical components of drag deceleration (D1 and D2, respectively) are obtained by projecting w_D along the appropriate coordinate axis.

(c) The effect of cross winds acting on the missile is modeled by a disturbance acceleration with the waveform

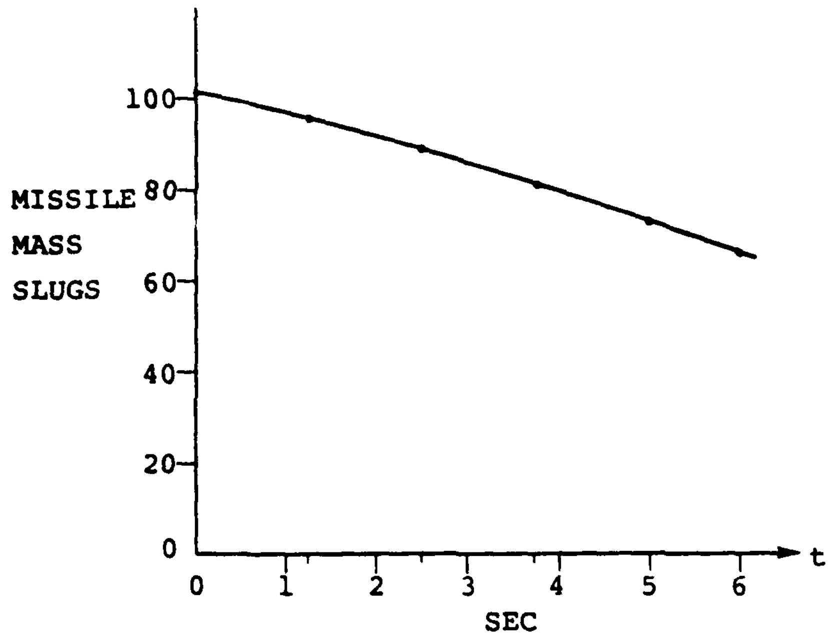


Figure 4-3. Missile mass versus burn time.

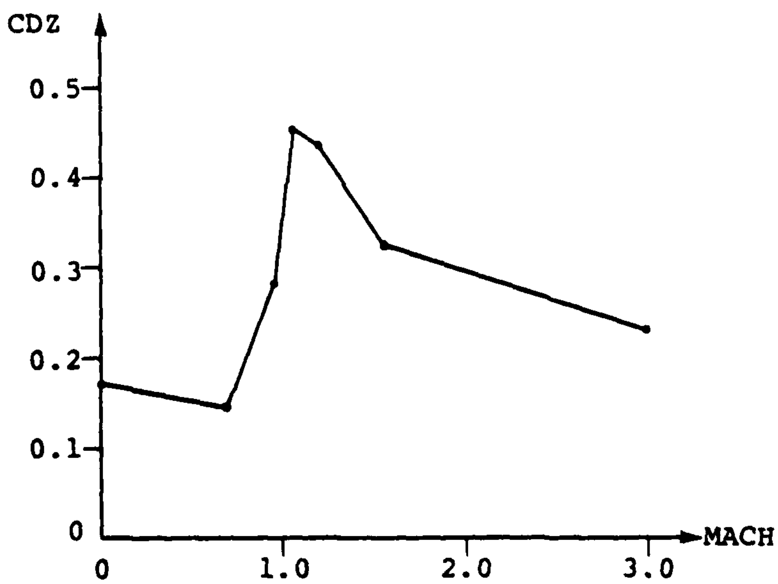


Figure 4-4. Missile drag coefficient versus mach number.

in Figure 4-5, acting in a direction normal to the missile velocity vector. This waveform is based on realistic aerodynamic performance data obtained in response to a wind force acting on a thrusting missile. Although the point-mass model used in the problem at hand lacks an explicit model of angle-of-attack (the angle between the missile longitudinal axis and its velocity vector is assumed to be zero), the wind model used in this problem simulates the effect of a local updraft or downdraft affecting a missile by causing it to develop a small angle-of-attack, which produces an aerodynamic force normal to the missile's longitudinal axis, which, in turn, causes an acceleration as shown in Figure 4-5. This acceleration is applied along a direction normal to the missile velocity vector. The duration of the acceleration (modeled as a sine-squared function with a peak value of 32.2 ft/sec^2) is based on the approximate time for an aerodynamically stable missile

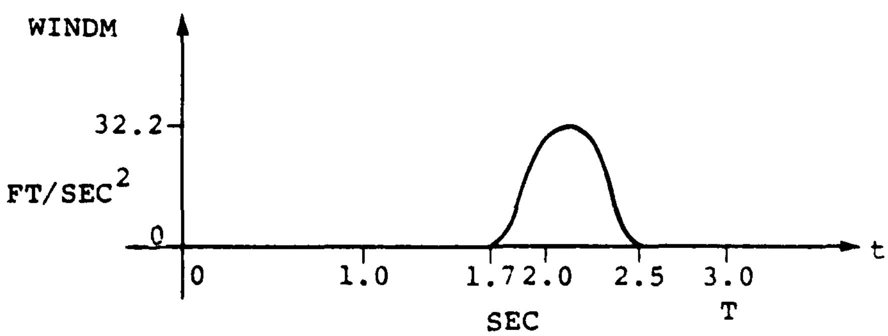


Figure 4-5. Representation of effect of wind disturbance on missile (WINDM).

(of the class considered in this problem) to regain its rotational balance following development of a wind-induced angle-of-attack. The direction of the acceleration WINDM is such as to move the missile <u>into</u> the wind for the type of thrusting missile assumed in this problem. The <u>translational</u> effect of winds occurring during thrusting, as distinguished from the rotational effect just described, results in a very small amount of translational motion, and can be neglected during the short flight times considered here. It is assumed in this problem that the wind effect is <u>local to the missile</u>, and does not affect the target.

(d) Intentional target maneuvers are modeled as a disturbance acceleration of the target resulting from aerodynamic forces acting in a direction normal to the target's velocity vector. It is assumed in this problem that the target develops sufficient thrust to just cancel its aerodynamic drag, and that the target's lift force is maintained at a level to balance the gravity force on the target; therefore, the target, in the absence of intentional maneuvers, would maintain a constant velocity and altitude. The assumed target acceleration, normal to its velocity vector, is shown in <u>Figure 4-6</u>. The maximum value of the latter target maneuver acceleration is 128 ft/sec^2, and its direction is represented as an uncertain parameter in the present study.

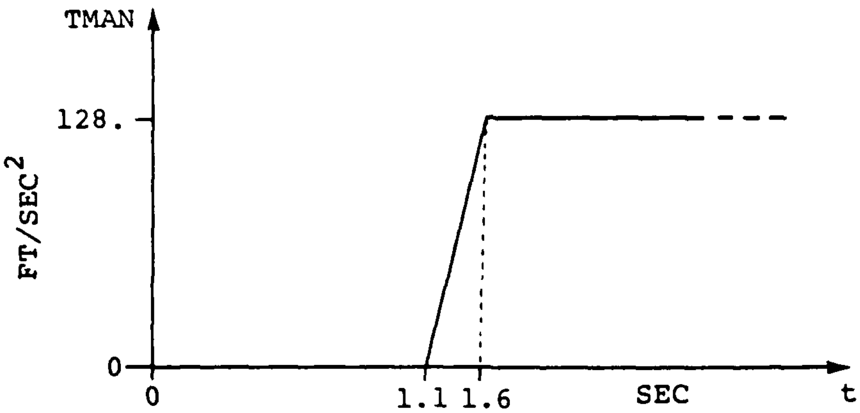


Figure 4-6. Target maneuver acceleration disturbance (TMAN).

4.2.3.1. Subcase 4.2.1 - Planar-Motion Intercept. Disturbance Inputs: (a) Gravity (helpful); (b) Aerodynamic drag (non-helpful). This Sub-case considers the performance of a missile under disturbance-utilizing control in a planar intercept problem with the particular missile-target geometry of Figure 4-7. This geometry represents the case in which the missile has been delivered, by a previous mid-course guidance phase, to the position (-9000 ft ground-range, +5000 ft altitude), with a horizontal velocity vector having a magnitude of 2000 ft/sec. The parameter values assumed for this Sub-case are:

(1)	Initial target ground-range	0. ft
(2)	Initial target altitude	4000. ft
	Initial target velocity*	-1000. ft/sec

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*... orizontal, to the left in Figure 4-7

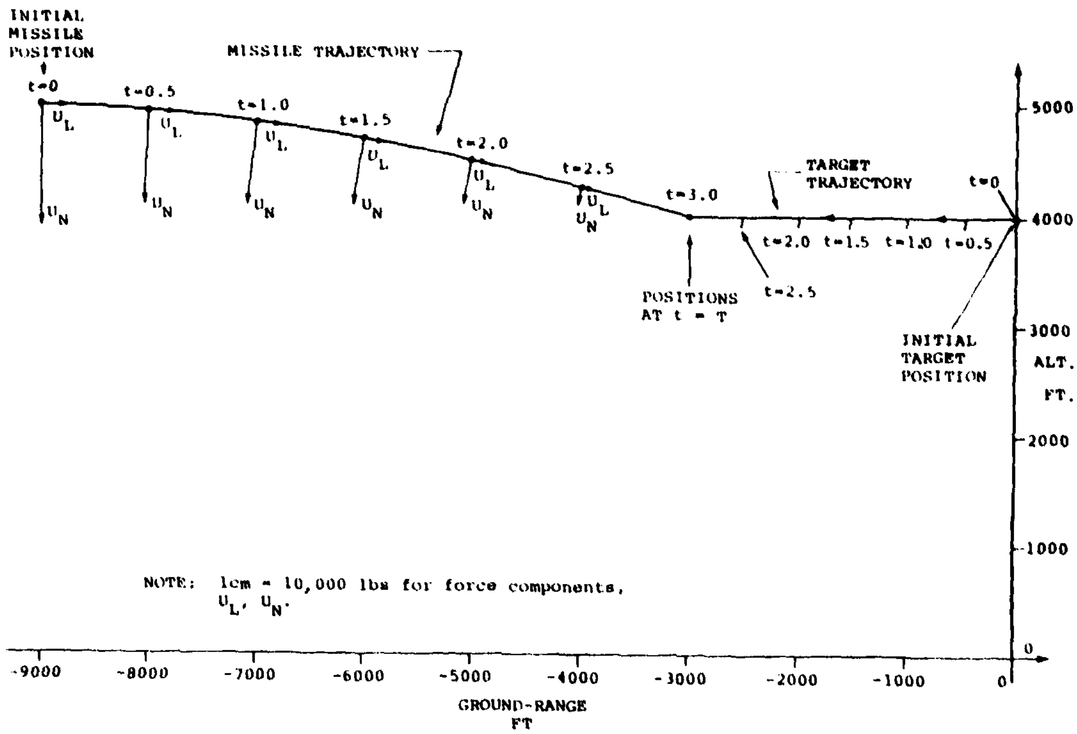

Figure 4-7. Missile and target trajectories for Sub-case 4.2.1, showing longitudinal (U_L) and normal (U_N) disturbance-utilizing control components. Disturbance inputs: (a) gravity; (b) aerodynamic drag.

NOTE: 1 cm = 10,000 lbs for force components U_L, U_N.

(4) Initial missile ground-range -9000. ft

(5) Initial missile altitude 5000. ft

(6) Initial missile velocity* 2000. ft/sec

(7) Disturbances present: gravity and aerodynamic missile drag (no target maneuver, no winds). Gravity is helpful (aids the intercept) and drag is counterproductive (acts against the intercept).

(8) Terminal state weighting matrix:

$$S = \begin{bmatrix} 10 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 10 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(9)

$$Q = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

(10)

$$r = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

(11) Specified terminal time = 3.0 sec.

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*Initially horizontal to the right in Figure 4-7

The resulting missile and target trajectories are presented in Figure 4-7. The optimal disturbance utilizing control forces can be represented by the two components U_L and U_N defined as follows:

U_L = control force along the longitudinal missile axis, computed as the projection of the optimal control forces mu_1 and mu_2.

U_N = control force normal to the longitudinal missile axis, computed as the projection of the optimal control forces mu_1 and mu_2.

A plot of $U_L(t)$ and $U_N(t)$, shown at half-second intervals, is depicted in Figure 4-7.

The longitudinal (U_L), normal (U_N), and resultant

$$U_{RES} = \sqrt{U_L{}^2 + U_N{}^2}$$

control forces for Sub-case 4.2.1 are plotted in Figure 4-8. Note that the primary control requirement is for normal (pitch-down) force in this case. The control forces in the relative-state coordinate system are plotted in Figure 4-9, showing that the primary control force requirement is in the vertical (downward) direction. The horizontal and vertical state histories for Sub-case 4.2.1 are shown in Figures 4-10 and 4-11. The disturbance histories for w_1 and w_2, reflecting the contributions of drag and gravity to the horizontal and vertical disturbances, respectively, are shown in Figure 4-12. Figure 4-13 shows a plot of the utility function $\mathcal{U}$, which remains positive for the entire control interval $[0 \leq t < T]$. Although drag is acting against achievement of the intercept, gravity is acting as a helpful disturbance during the interval $[t_o, T]$, resulting in a net disturbance effect which is helpful.

The performance Sub-case 4.2.1 is summarized in Table 4-1 where the disturbance-utilizing controller (DUC), using the control law

$$u^o = - R^{-1}B^T \left[K_x x + K_{xz} z\right] \tag{4.45}$$

is compared with the so-called "conventional linear-quadratic controller," called the "LQ" controller. The LQ controller uses the familiar control law

$$u = - R^{-1}B^T \left[K_x x\right] \tag{4.46}$$

which is the same as Equation (4.45) with $K_{xz} \equiv o$. Except for the difference in control laws, Equations (4.45) and (4.46), the conditions under which the comparison is made are identical (i.e., identical disturbance inputs, identical plants, identical geometry, etc.).

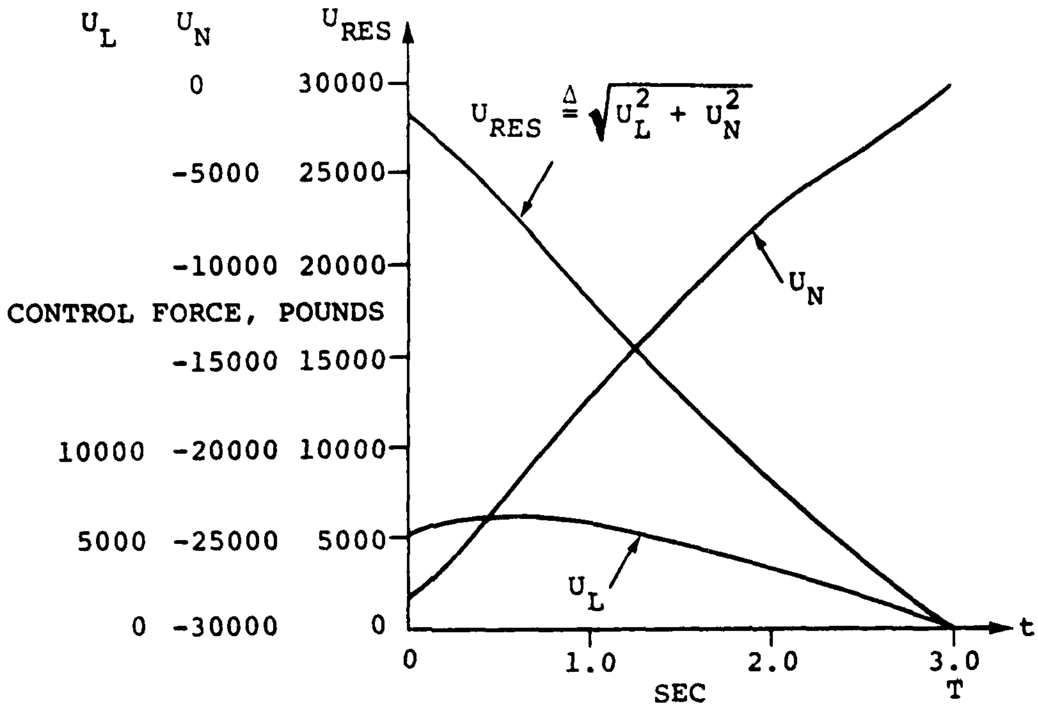

Figure 4-8. Longitudinal (U_L), Normal (U_N), and Resultant (U_{RES}) of control force for Sub-case 4.2.1; disturbance-utilizing control.

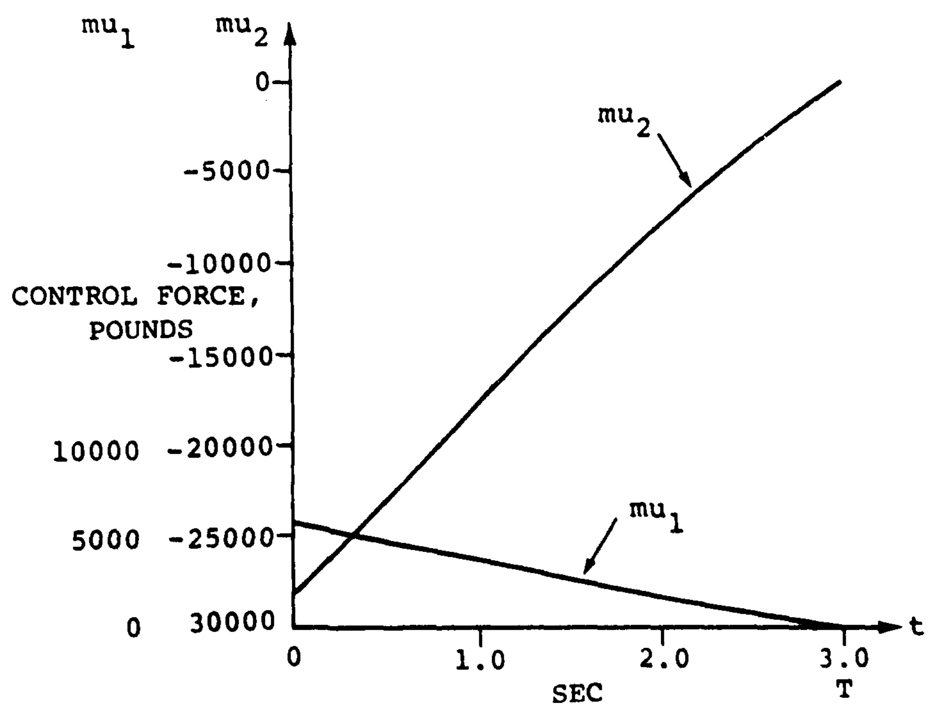

Figure 4-9. Horizontal (mu_1) and Vertical (mu_2) control forces for Sub-case 4.2.1; disturbance-utilizing control.

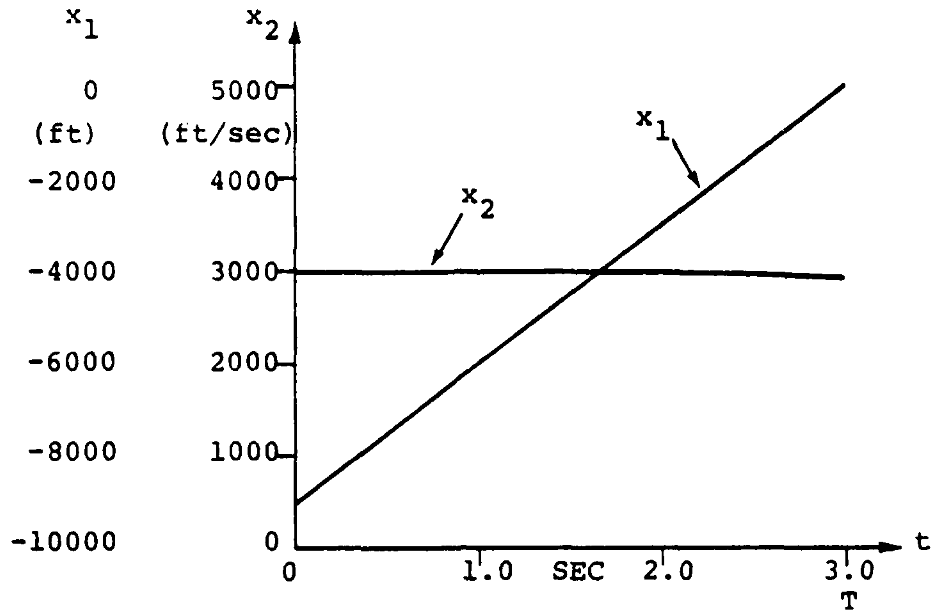

Figure 4-10. Horizontal state histories: x_1 (relative position), x_2 (relative velocity), for Sub-case 4.2.1; disturbance-utilizing control.

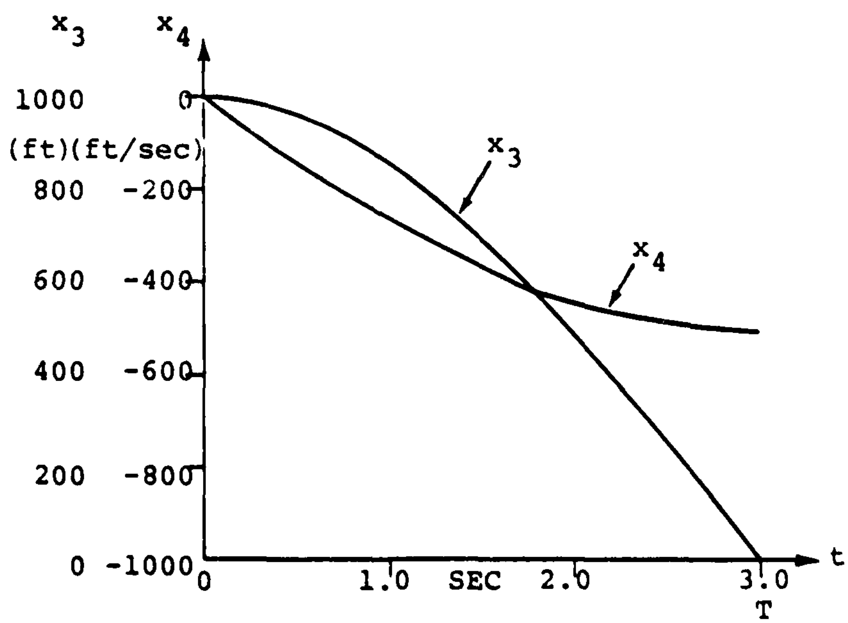

Figure 4-11 Vertical state histories: x_3 (relative position), x_4 (relative velocity), for Sub-case 4.2.1, disturbance-utilizing control.

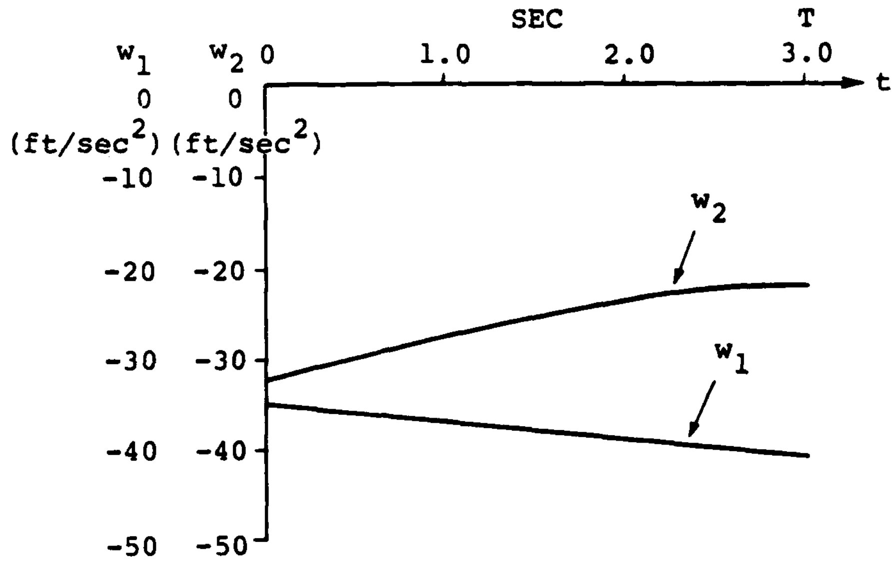

Figure 4-12. Relative accelerations due to disturbances: w_1 (horizontal) and w_2 (vertical), for Sub-case 4.2.1.

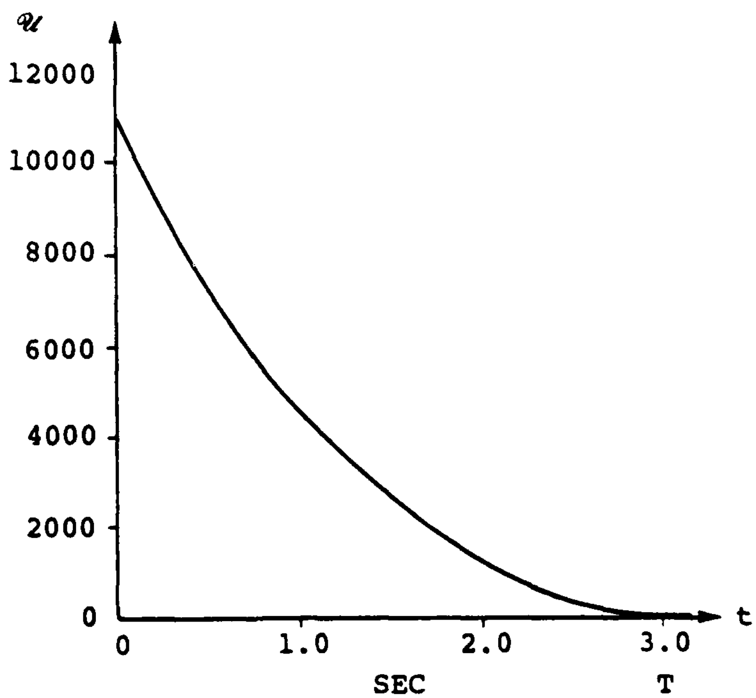

Figure 4-13. Disturbance utility for Sub-case 4.2.1.

The various performance measures presented in Table 4-1 are defined as follows:

Table 4-1. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.2.1.

	PERFORMANCE INDEX J	$\mathcal{E}_T$ %	CONTROL ENERGY EU	CONTROL FUEL EAU	HORIZ. MISS x_1 (T) (FT)	$\mathcal{E}_{M_H}$ %	VERT. MISS x_3 (T) (FT)	$\mathcal{E}_{M_V}$ %	MISS DISTANCE MD (FT)	$\mathcal{E}_{MD}$ %
DUC	0.435 $\times 10^5$	9.6	0.430 $\times 10^5$	277.0	-1.9	89.0	9.5	-72.1	9.7	43.9
LQ	0.481 $\times 10^5$		0.466 $\times 10^5$	274.0	-17.3		0.13		17.3	

NOTE: SEE PAGE 157 FOR DEFINITIONS OF J, $\mathcal{E}_T$, EU, EAU, $\mathcal{E}_{M_H}$, $\mathcal{E}_{M_V}$, MD AND $\mathcal{E}_{MD}$.

$$J = \frac{1}{2} x^T(T) S x(t) + \frac{1}{2} \int_0^T \left[x(t)^T Q x(t) + u^T(t) R u(t) \right] dt \tag{4.47}$$

$$\mathscr{E}_T = \frac{J_{K_{xz}=0} - J_{K_{xz}\neq 0}}{J_{K_{xz}=0}} \times 100\% \tag{4.48}$$

$$EU = \frac{1}{2} \int_{t_0}^{T} u^T(t) u(t) \quad dt \tag{4.49}$$

$$EAU = \frac{1}{2} \int_{t_0}^{T} \left(|u_1(t)| + |u_2(t)| \right) dt \tag{4.50}$$

$$\mathscr{E}_{M_h} = \frac{|x_1(T)_{LQ}| - |x_1(T)_{DUC}|}{|x_1(T)_{LQ}|} \times 100\% \tag{4.51}$$

$$\mathscr{E}_{M_v} = \frac{|x_3(T)_{LQ}| - |x_3(T)_{DUC}|}{|x_3(T)_{LQ}|} \times 100\% \tag{4.52}$$

$$MD = \sqrt{x_1^2(T) + x_3^2(T)} \tag{4.53}$$

$$\mathscr{E}_{MD} = \frac{(MD)_{LQ} - (MD)_{DUC}}{(MD)_{LQ}} \times 100\% \tag{4.54}$$

The measure EU, computed by Equation (4.49), is proportional to the "control energy" part of the integral term in the performance index J (Equation 4.47). The measure EAU, computed by Equation (4.50) is of interest to control actuator designers, since it gives an indication of how much actuator fuel, or control thruster fuel, is consumed during a missile flight.

For Sub-case 4.2.1, the DUC controller achieves a smaller performance index J (evidenced by a positive value of δ_T). This is the expected result, since the LQ controller does not have benefit of information about the disturbance vector z and does not incorporate the time-varying gain matrix K_{xz}. The DUC also achieved a substantially smaller miss distance MD (evidenced by the large positive value of δ_{MD}), even though the LQ controller achieved a smaller vertical miss distance (note the negative value of δ_{M_V}). The DUC required less control energy EU, but slightly more control fuel EAU in this flight. Thus, the impressively smaller miss distance MD was achieved at the expense of a small additional expenditure of fuel (relative to the LQ controller) for this sub-case. Heavier weighting on the matrix S might result in more efficient fuel consumption for the DUC in this particular sub-case.

The performance cited for Sub-case 4.2.1 is based on the specified terminal time T = 3.0 seconds. It is of interest to examine the effects of using other fixed values of T. Based on an approach suggested by Prof. C. D. Johnson, a digital computer program was developed to determine the value of J for a range of values of T. This program scans the set of values of J thus obtained, selects the minimum value J_{min} among them, and prints the results of the optimal control problem which has the particular terminal time T_{min} which yields J_{min}. The program first solves the matric differential equations for K_x, K_{xz} and K_z (Equations 4.41, 4.42 and 4.43) <u>in backward time</u>, starting at the given terminal conditions. For example, if it is desired to scan over a range of values of T from T = 0 to T = 6 seconds, the matric equations will be solved beginning at 6 seconds, using the given terminal conditions, and integrating until t = 0 is reached. During this backward integration procedure, intermediate values of the gain matrix elements are stored at selected intervals. For instance, if an interval of 1 second is used, then values of each element are stored at t = 5, 4, 3, 2, 1 and 0 seconds. Figure 4-14 illustrates this procedure for one element ($K_{x_{11}}$) of the 16 element K_x array for Sub-case 4.2.1. For this element, the values $K_{x_{11}}(5)$, $K_{x_{11}}(4)$, $K_{x_{11}}(3)$, $K_{x_{11}}(2)$, $K_{x_{11}}(1)$ and $K_{x_{11}}(0)$ are saved during backward integration of the matric differential equation. Next, the optimal control problem is solved in <u>forward-time</u>, using each saved value as an initial condition for a forward-time integration of the matric differential equations. In the example being considered, six forward-time optimal control problems are solved, and J is computed for each problem.

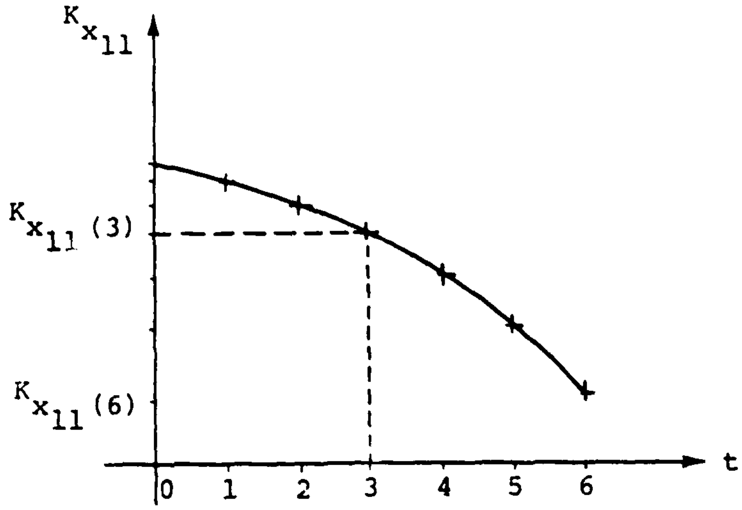

Figure 4-14. Typical history of $K_{x_{11}}$.

The program selects the minimum (J_{min}) of these six values (if a minimum exists) and identifies the corresponding terminal time T_{min}. As an example, suppose the minimum value of J is found to be associated with $T_{min} = 3$ seconds; then, the program makes one final "run for the record", using $T = T_{min} = 3$ seconds and the initial conditions $K_x(3)$, $K_{xz}(3)$ and $K_z(3)$. (The matrices K_x and K_{xz} are required for the control law implementation; matrix K_z is used in computing the utility function, for analysis purposes). For the gain element $K_{x_{11}}$, the resulting gain history, generated by this final forward-time integration appears as shown in Figure 4-15. In addition to providing the detailed data for the J_{min} case, the program also outputs summary parameters such as J, terminal-time miss distance $MD = \sqrt{x_1^2(T) + x_3^2(T)}$

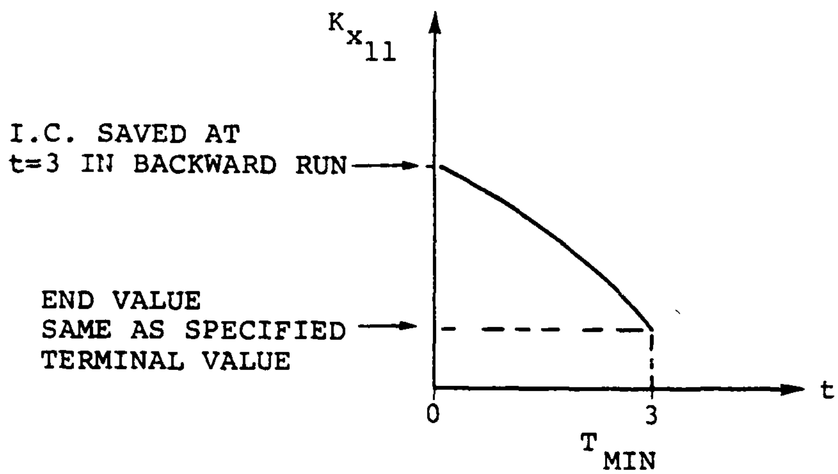

Figure 4-15. Forward-time history of $K_{x_{11}}$ for T_{min}=3.

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EU and EAU for each value of T in the range scanned. Values of these parameters for various values of T, for the problem conditions of Sub-case 4.2.1, are presented in Figures 4-16 through 4-19 for the disturbance-utilizing controller, and the conventional LQ controller. The minimum value of J is reached at T = 3.0 seconds for both controllers (see Figure 4-16). The total effectiveness $\mathcal{E}_T$ versus T is plotted over the range of T from 1.5 to 5.0 seconds (Figure 4-20), remaining positive for the whole interval. This shows that the disturbance-utilizing controller consistently obtains a lower performance index than the LQ controller.

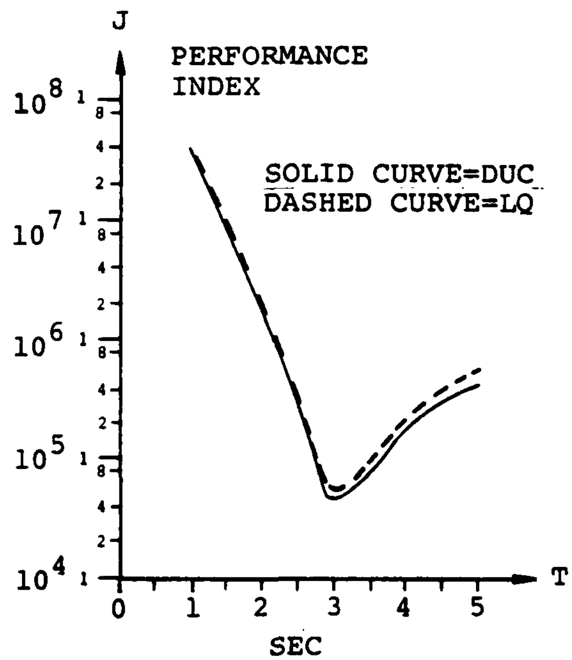

Figure 4-16. Performance index J versus terminal time T, Sub-case 4.2.1; DUC and LQ.

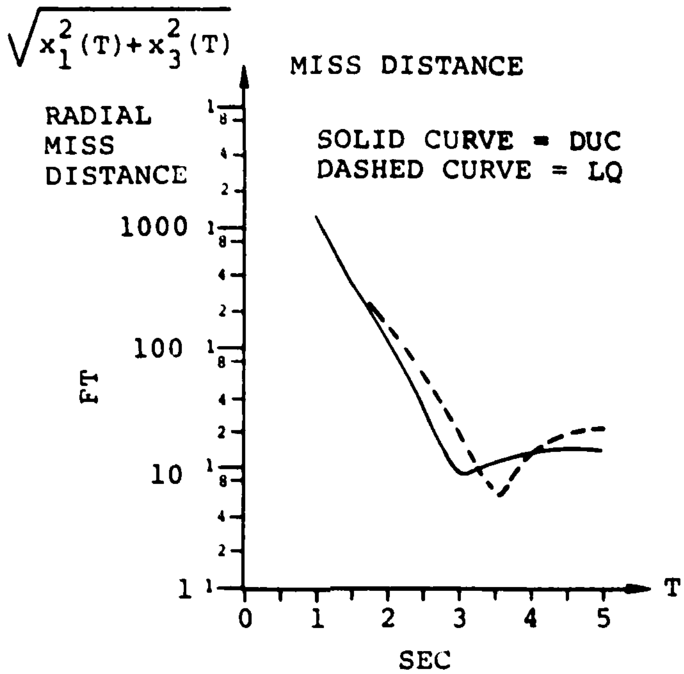

Figure 4-17. Radial miss distance at t = T versus terminal time T, Sub-case 4.2.1; DUC and LQ.

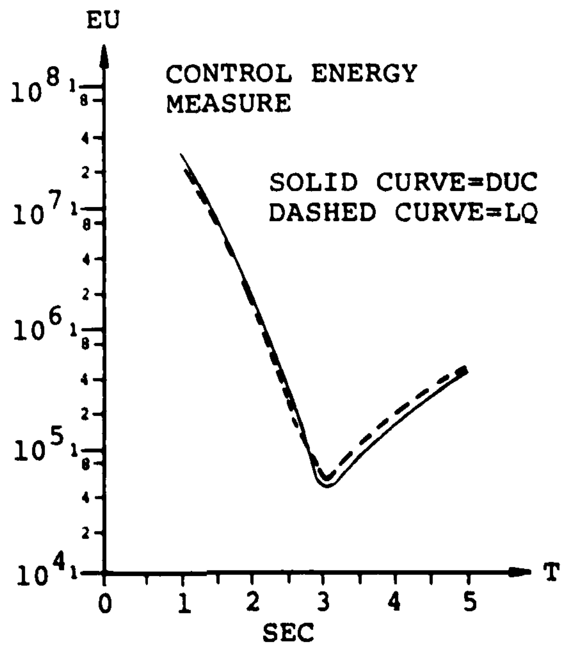

Figure 4-18. Control energy measure EU versus terminal time T, Sub-case 4.2.1; DUC and LQ.

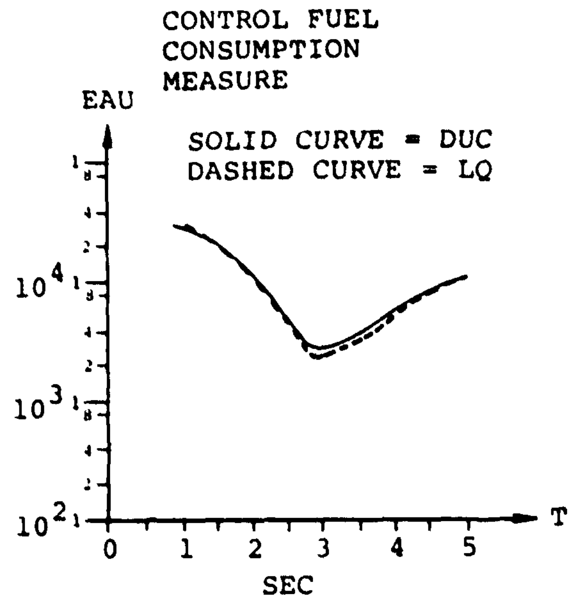

Figure 4-19. Control fuel measure EAU versus terminal time T, Sub-case 4.2.1; DUC and LQ.

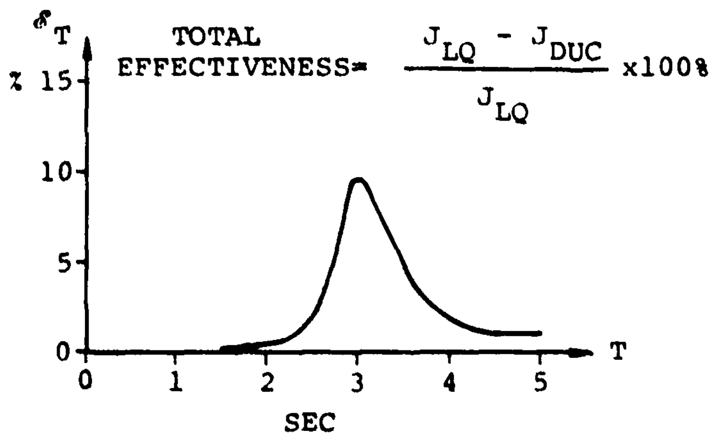

Figure 4-20. Total effectiveness $\mathscr{E}_T$ versus specified terminal time T for Sub-case 4.2.1.

4.2.3.2 Sub-case 4.2.2 Planar - Motion Intercept with Disturbances Inputs:

a) Gravity (helpful)

b) Aerodynamic drag (non-helpful)

c) Winds (non-helpful in vertical direction; helpful in horizontal direction) on the missile

d) Target maneuver (non-helpful)

This case considers the effects of winds and target maneuvers on the performance of a missile with disturbance-utilizing control. All conditions and parameters in this case are identical with those of Sub-case 4.2.1, except that two additional disturbance inputs are present in sub-case 4.2.2: namely, winds and target maneuvers. The wind input disturbance is defined by Figure 4-5, and acts on the missile as a normal force (m)(WINDM), where m is the missile mass and WINDM is the normal acceleration of the missile due to wind. It is assumed in Sub-case 4.2.2 that the wind force (m)(WINDM), or its horizontal and vertical components (m)(WIND1) and (m)(WIND2), are acting on the missile as shown in Figure 4-21. Thus, for the geometry of Sub-case 4.2.2, the horizontal component of wind force is helpful in driving the missile toward the intercept, but the vertical component of wind force is non-helpful, since it acts to hinder the intercept.

The target maneuver in Sub-case 4.2.2 has the waveform described in Figure 4-6, with a peak acceleration value of 128. ft/sec^2 (4g), and acting in a direction to drive the

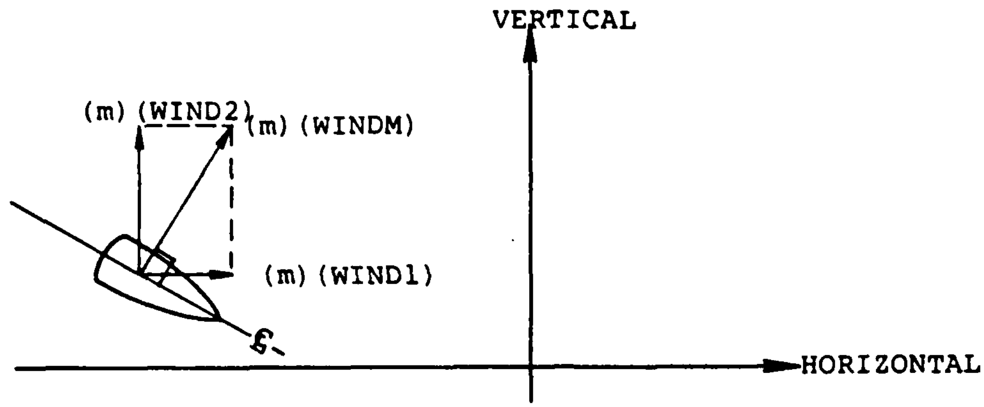

Figure 4-21. Normal wind force (m)(WINDM) and its horizontal and vertical components (m)(WIND1), (m)(WIND2), respectively, for Sub-case 4.2.2.

target downward---i.e., away from intercept. Thus a non-helpful target maneuver is assumed.

The closed-loop disturbances-utilizing control was derived as in previous cases, and the resulting missile and target trajectories are shown in Figure 4-22. The longitudinal and normal components of the disturbance-utilizing control force are also shown in Figure 4-22 at selected moments of time. Note the downward deflection of the target trajectory due to the assumed target acceleration maneuver. Although the target applies a significant magnitude of maneuver acceleration (4g), it is applied about half-way into the flight and the target trajectory does not have time to deviate very far before intercept occurs. The effect of the target maneuver "disturbance" on the components U_L and U_N of the disturbance-utilizing control force can be seen at $t = 1.5$ in Figure 4-22, where, for

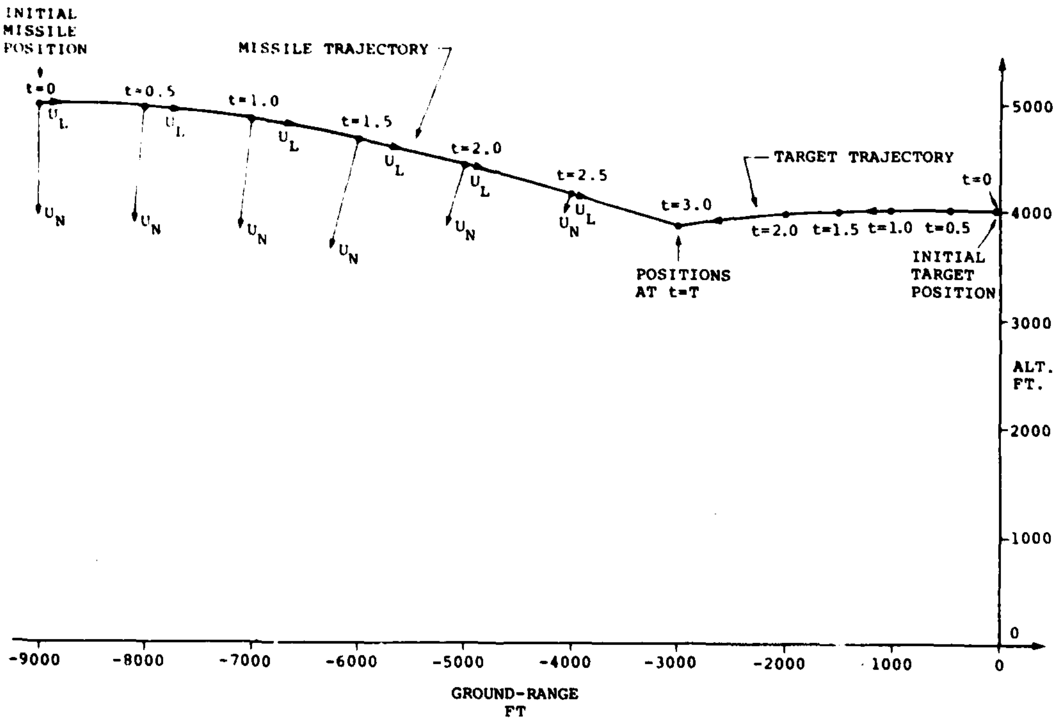

Figure 4-22. Missile and target trajectories for Sub-case 4.2.2 showing longitudinal (U_L) and normal (U_N) disturbance-utilizing control force components. Disturbance inputs:

a) gravity
b) aerodynamic drag
c) winds on missile
d) target maneuver

Note: 1cm = 10,000 lbs for force components U_L, U_N.

instance, both control components become larger for a short time to accommodate the detrimental effects of the target maneuver "disturbance".

The time-histories of the disturbance-utilizing control forces for Sub-case 4.2.2 (Figures 4-23 and 4-24) clearly show how the additional disturbance effects due to target maneuvers and winds are taken into account by the disturbance-utilizing controller. Note, for example, the abrupt changes in the control forces over the sub-interval [t = 1.1 to 1.6 sec] due to the onset of target maneuvers. The effect of the wind on the control forces can likewise be seen during the sub-interval [t = 1.7 to 2.5 sec].

The time-histories of the relative state variables (Figures 4-25 and 4-26) are almost identical with the corresponding time-histories for sub-case 4.2.1 (Figures 4-10 and 4-11), which indicates that the disturbance-utilizing controller is doing an effective job of "accommodating" the additional disturbance inputs due to target maneuvers and winds.

The relative accelerations between missile and target, due to disturbances, are shown in terms of horizontal (w_1) and vertical (w_2) components in Figure 4-27. Those graphs show the rather dramatic changes in relative accelerations due to the presence of additional disturbance inputs for this sub-case.

The effects on performance caused by the additional disturbances can also be seen in the time-history of utility

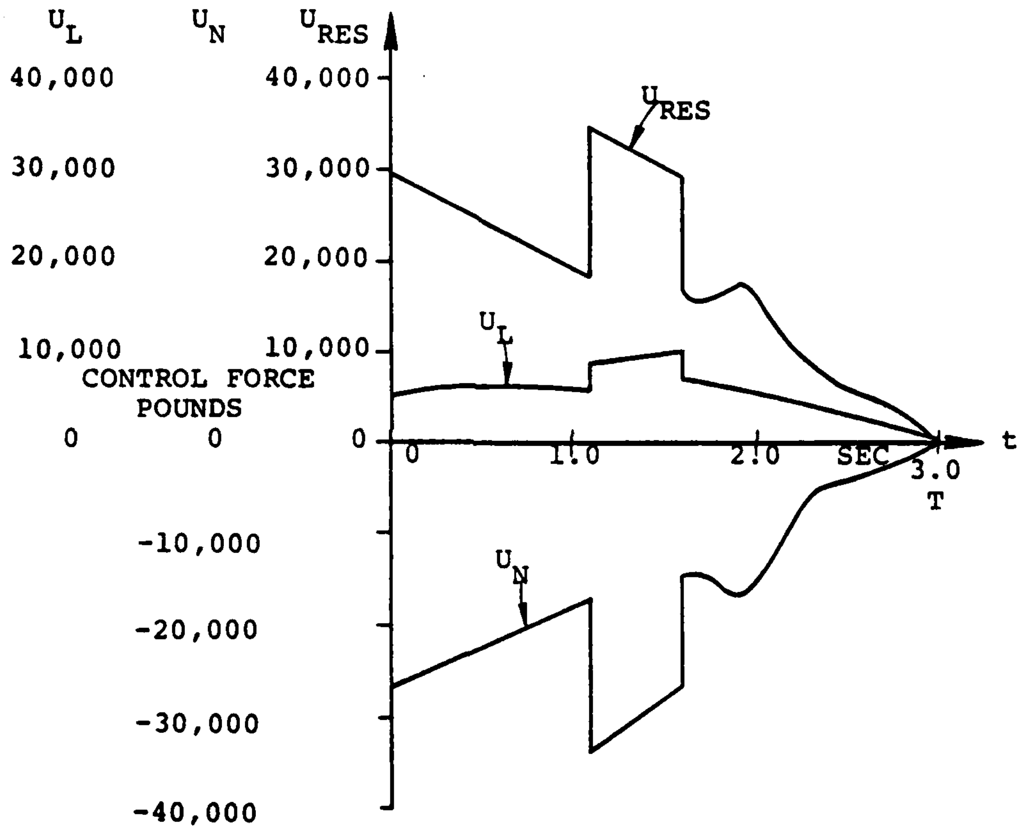

Figure 4-23. Longitudinal (U_L), Normal (U_N), and Resultant (U_{RES}) of control force for Sub-case 4.2.2; disturbance-utilizing control.

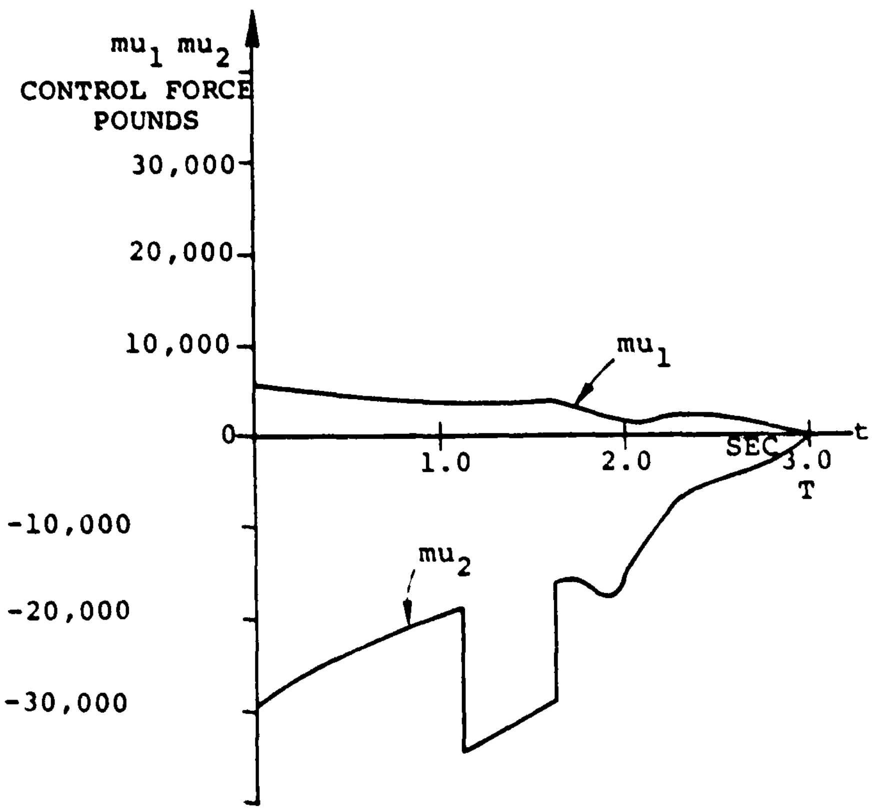

Figure 4-24. Horizontal (mu_1) and Vertical (mu_2) control forces for Sub-case 4.2.2; disturbance-utilizing control.

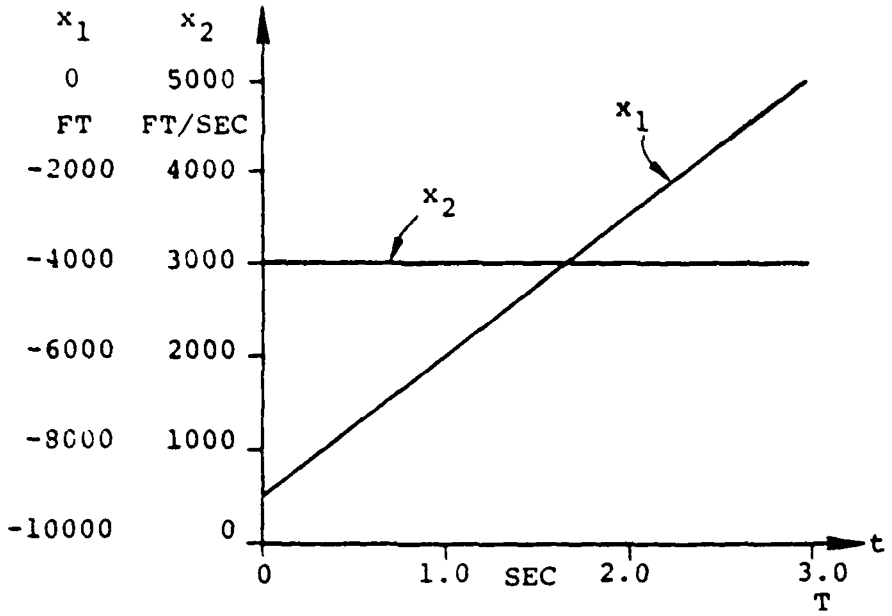

Figure 4-25. Horizontal state histories: x_1 (relative position), x_2 (relative velocity), for Sub-case 4.2.2; disturbance-utilizing control.

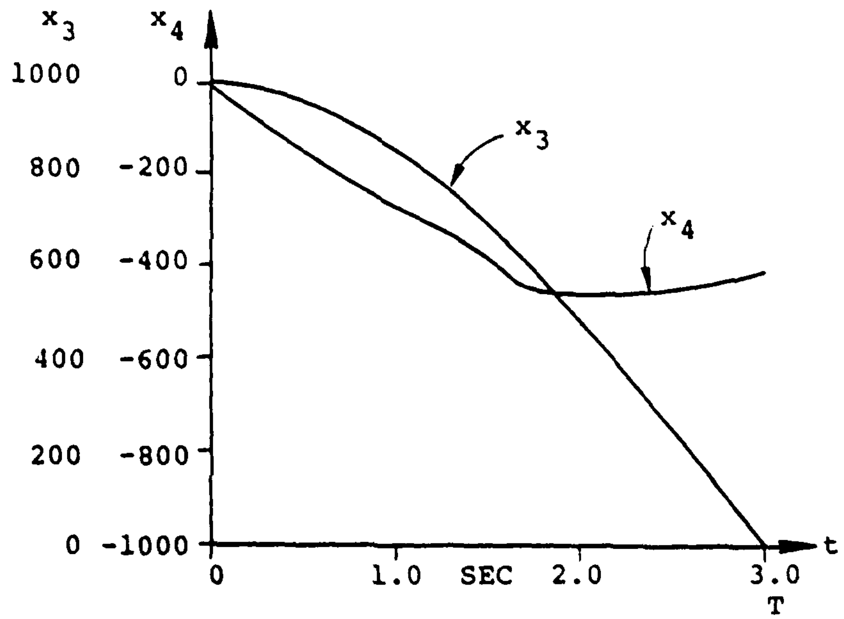

Figure 4-26. Vertical state histories: x_3 (relative position), x_4 (relative velocity), for Sub-case 4.2.2; disturbance-utilizing control.

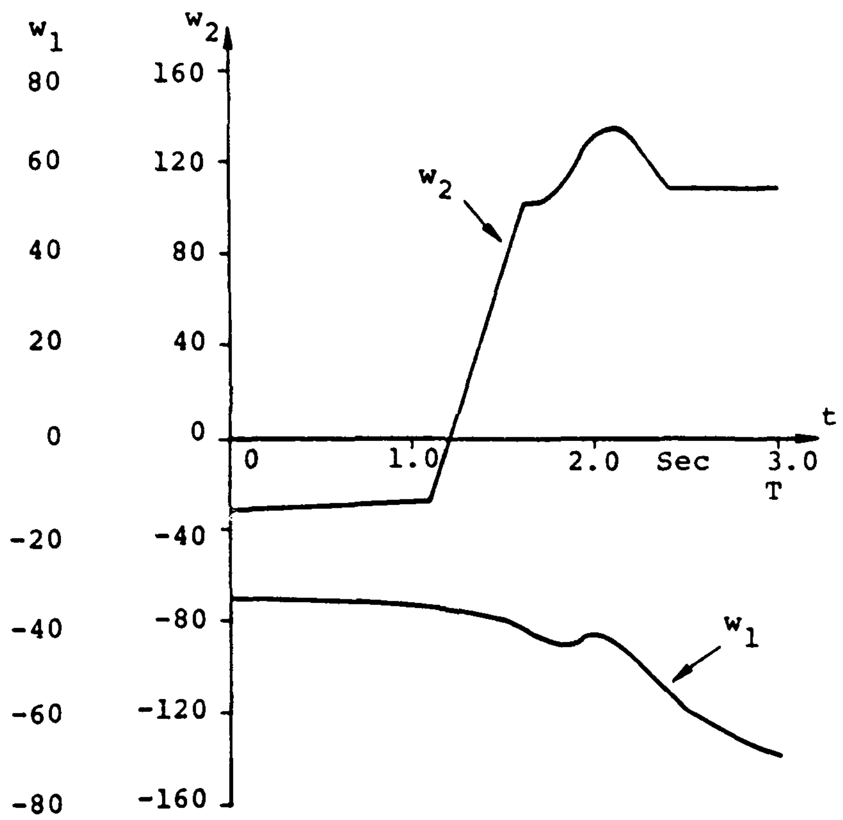

Figure 4-27. Relative accelerations due to disturbances: w_1 (horizontal) and w_2 (vertical), for Sub-case 4.2.2.

(Figure 4-28) which is suddenly driven to a large negative value by the target maneuver input, beginning at 1.1 seconds. Thus, this sub-case is seen to have a considerably less favorable utility history than was true for sub-case 4.2.1.

The performance summary for Sub-case 4.2.2 is presented in Table 4-2. The disturbance-utilizing controller for this case obtains a larger J than for Sub-case 4.2.1, reflecting primarily the increased control energy (as measured by EU) required to accommodate the intensified disturbance environment. It is interesting to note from Table 4-2 that, although the disturbance-utilizing controller achieves both a lower J and a lower terminal miss-distance than the conventional LQ controller, it did not in this case use less control energy as measured by EU. This is not a surprising occurrence, since the structure of the performance index J, Equation (4.39), leads to an optimal solution which seeks a weighted balance between the values of the terminal state term and the value of the integral portion of J. In Sub-case 4.2.2, the terminal miss-distance performance of the DUC was achieved at the expense of consuming a small amount of additional control energy. However, the DUC consumes less fuel than the LQ controller, for this sub-case as shown by the values of EAU in Table 4-2. Note that the disturbance-utilizing controller has positive total effectiveness , $\mathscr{E}_T = 11.2\%$, for the selected terminal time of $T = 3.0$ sec. The variation of $\mathscr{E}_T$ versus various specified

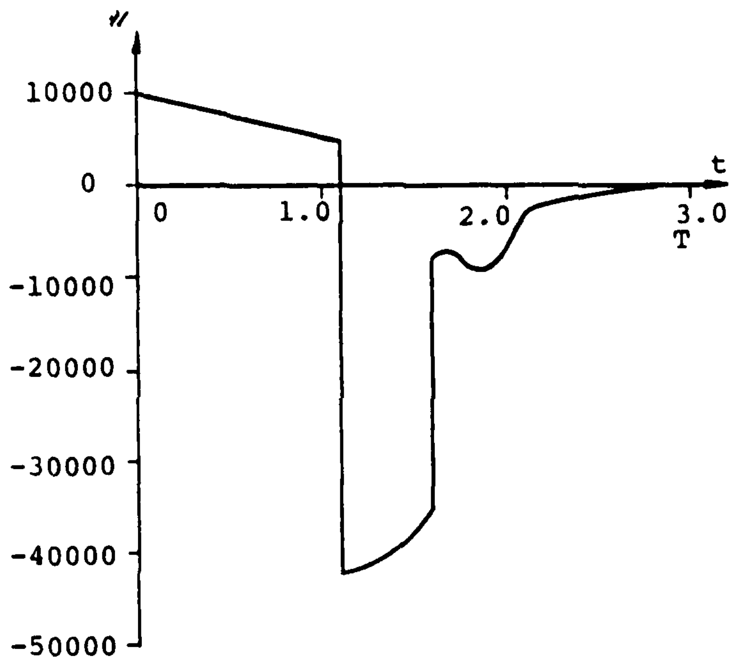

Figure 4 - 28. Disturbance utility for Sub-case 4.2.2.

Table 4-2. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.2.2.

	PERFORMANCE INDEX J	ϵ_T %	CONTROL ENERGY EU	CONTROL FUEL EAU	HORIZ. MISS $x_1(T)$ (FT)	ϵ_{M_H} %	VERT. MISS $x_3(T)$ (FT)	ϵ_{M_V} %	MISS DISTANCE MD (FT)	ϵ_{MD} %
DUC	0.734×10^5	11.2	0.727×10^5	371.0	-3.9	82.2	10.5	79.0	11.2	79.5
LQ	0.827×10^5		0.678×10^5	387.0	-21.9		50.0		54.6	

NOTE: SEE PAGE 157 FOR DEFINITIONS OF J, ϵ_T, EU, EAU, ϵ_{M_H}, ϵ_{M_V}, MD AND ϵ_{MD}.

terminal times is shown in Figure 4-29. Note that $\mathscr{E}_t$ remains positive up to about 4.7 seconds. The effectiveness would be expected to be always non-negative (that is, the disturbance-utilizing controller would be expected to be at least as good as the conventional linear-quadratic controller in handling an identical disturbance input, but, the numerical results of Figure 4-29 indicate that this condition was not achieved beyond t = 4.7 seconds for the calculations performed on Sub-case 4.2.2. The reason(s) for this discrepancy is not known, but the following two possible sources of error are proposed as contributing factors:

(1) Disturbance waveforms consisting of linear ramps and constant level segments (as assumed in the disturbance model, Equation (4.28) - (4.32)) only afford an approximation to the wind and drag disturbances actually used as inputs in the computer simulation runs of this problem. The non-ideal results in the values of $\mathscr{E}_T$ above t = 4.7 may reflect the error involved in this approximation since $\mathscr{E}_T$ is a function of the difference between two large but nearly equal, values of J in this region. A solution to this problem, of course, is to model the disturbance more realistically, possibly increasing the order of the disturbance model, Equations (4.28) - (4.32). On the other hand, the level of the existing approximation error may well be adequate for most practical intercept applications.

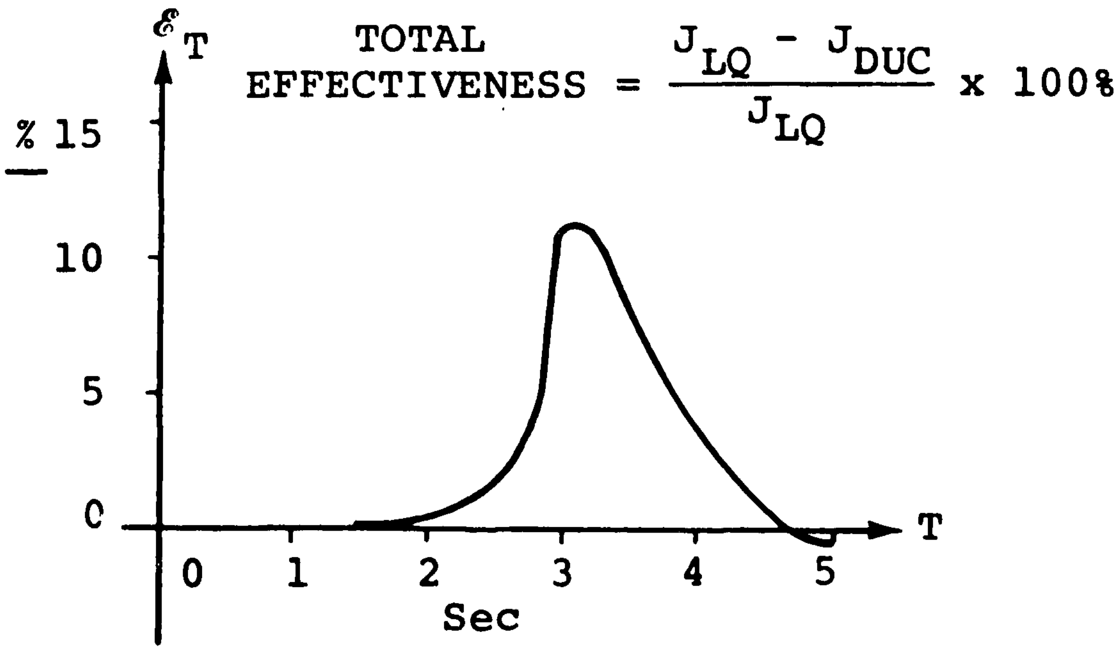

Figure 4-29. Total effectiveness $\mathscr{E}_T$ versus specified terminal time T for Sub-case 4.2.2.

(2) The method of obtaining "initial conditions" for the forward-time computation of the gain matrices K_x and K_{xz}, as used in this study, may introduce computational errors in cases with large values of T. That is, since some elements of the matrices are very "flat" as backward-time approaches t = o, the backward-time integration process may produce neighboring solutions having nearly identical values near t = o. Figure 4-30 illustrates this phenomenon for a typical element of the matrix K_x, where several gain-histories have nearly the same initial conditions at t = o. In fact, due to the resolution limitation imposed by the digital word size, there always exists some "large" value of T such that two neighboring $K_x(t)$ gain-histories will have values so close together (near t = o) that their numerical values are represented by the same digital word. When this happens, it becomes impossible to generate unique forward-time gain-histories for two neighboring backward-time solutions. One way to circumvent this problem is to provide double-precision computations for this portion of the program.

4.2.3.3 Sub-case 4.2.3 - Planar - Motion Intercept. All Conditions as in Sub-case 4.2.2, Except for Increased Terminal State Weighting. It is interesting to consider the effects of the terminal state weighting matrix S on the performance of a missile with disturbance-utilizing control. The sub-case considered in this section uses the S-matrix

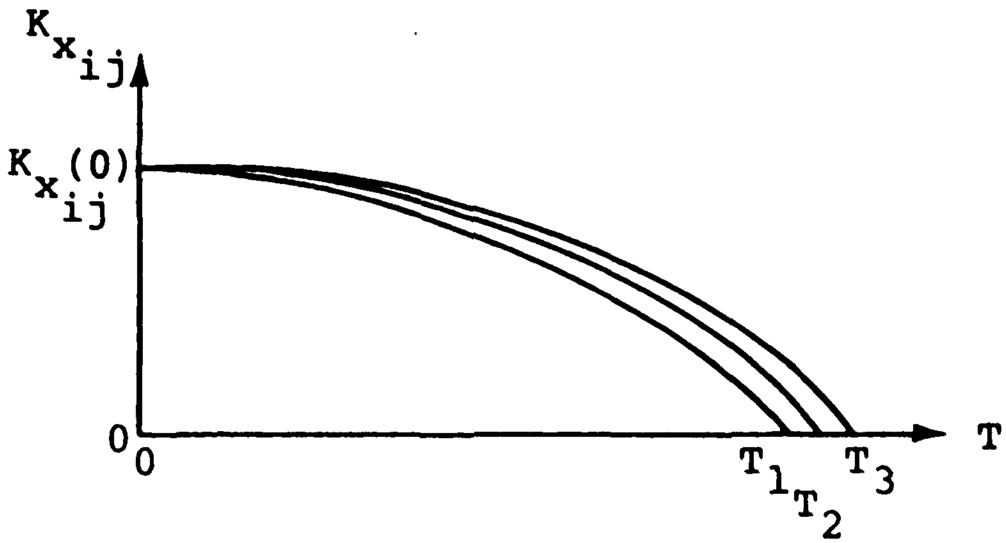

Figure 4-30. Illustration of problem of ambiguity due to "flatness" near t = o.

$$S = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad (4.55)$$

That is, the weighting of the relative position states x_1 and x_3 is increased by a factor of 10 over that used in Sub-case 4.2.2; all other parameters are identical with those of Sub-case 4.2.2.

The missile performance summary for sub-case 4.2.3. is presented in Table 4-3, and shows that the increased weighting on terminal miss-distance provides a sharp reduction in both $x_1(T)$ and $x_3(T)$, as well as total miss-distance, for both the disturbance-utilizing controller and the conventional linear-quadratic controller (compare with Table 4-2). These reductions were obtained at the expense of slightly larger total values of J. Note that, for Sub-case 4.2.3, the disturbance-utilizing controller achieves much lower terminal miss-distance and also uses less control energy and less control fuel than the conventional LQ controller. The disturbance-utilizing controller for this case achieves significant total effectiveness value $\mathscr{E}_T$ = 20.2%.* Figure 4-31 shows the total effectiveness $\mathscr{E}_T$ versus terminal time T for Sub-case 4.2.3, which indicates that the disturbance-utilizing controller consistently obtains a lower value of J for various values of T.

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*This means that J_{DUC} is 79.8% of J_{LQ}.

TABLE 4-3. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.2.3.

	PERFORMANCE INDEX J	ϵ_T %	CONTROL ENERGY EU	CONTROL FUEL EAU	HORIZ. MISS x_1 (T) (FT)	ϵ_{M_H} %	VERT. MISS x_2 (T) (FT)	ϵ_{M_V} %	MISS DISTANCE MD (FT)	ϵ_{MD} %
DUC	0.743×10^5	20.2	0.742×10^5	376.0	-0.6	90.2	0.9	92.4	1.1	91.7
LQ	0.930×10^5		0.842×10^5	443.0	-6.1		11.8		13.3	

NOTE: SEE PAGE 157 FOR DEFINITIONS OF J, ϵ_T, EU, EAU, ϵ_{M_H}, ϵ_{M_V}, MD and ϵ_{MD}.

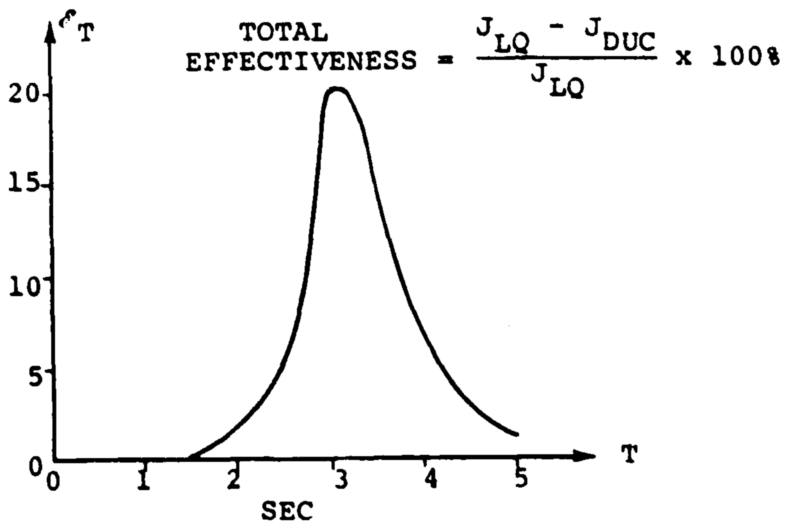

Figure 4-31. Total effectiveness ϵ_T versus specified terminal time T for Sub-case 4.2.3.

4.2.3.4 Sub-case 4.2.4 - Planar - Motion Intercept; Ground-Launched Missile. Disturbance Inputs:

a) Gravity (non-helpful)

b) Aerodynamic drag (non-helpful)

c) Winds (non-helpful in horizontal direction, helpful in vertical direction.)

d) Target maneuver (non-helpful)

In this section we consider the performance of a missile with disturbance-utilizing control in a ground-launch problem with difficult missile-target geometry and highly detrimental disturbance inputs. The geometry of the case to be considered is shown in Figure 4-32. The parameter values peculiar to Sub-case 4.2.4 are:

a)	Initial target ground-range	0. ft
b)	Initial target altitude	6000. ft
c)	Initial target velocity (horizontal, to the left in Figure 4-32)	-1000. ft/sec
d)	Initial missile ground-range	-7000. ft
e)	Initial missile altitude	0. ft
f)	Initial missile velocity	0. ft/sec

g) Disturbances present: gravity (non-helpful), aerodynamic drag (non-helpful), wind force on missile (large non-helpful component in horizontal direction; small helpful component in vertical direction), target maneuver (4 g's, evasive - non-helpful).

h) Terminal state weighting matrix S is as follows:

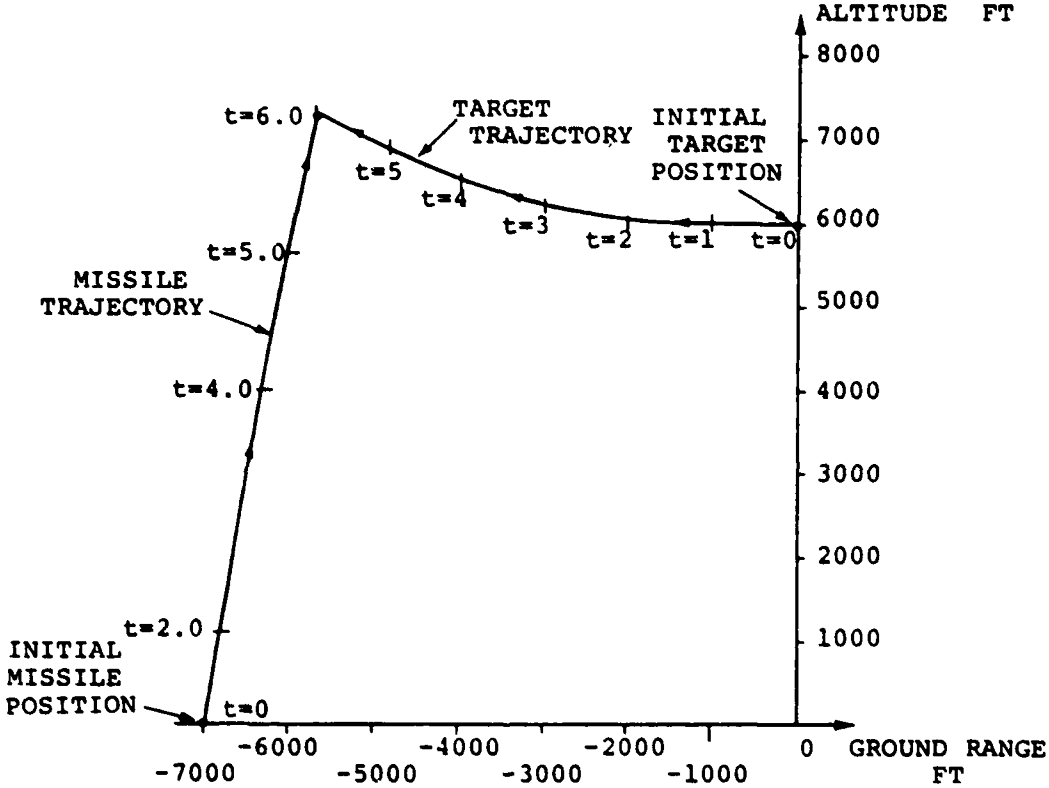

Figure 4-32. Missile and target trajectories for Sub-case 4.2.4; disturbance-utilizing control. Disturbance inputs:

a) Gravity

b) Aerodynamic drag

c) Wind on missile

d) Target maneuver

$$S = \begin{bmatrix} 100 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 100 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad (4.56)$$

i) $Q = [0]$ (4.57)

j) $R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ (4.58)

k) Specified terminal time T = 6.0 sec.

The wind disturbance force for Sub-case 4.2.4 is oriented as shown in Figure 4-33. The missile trajectory for this case is nearly vertical (about 80 degrees), and, therefore, the horizontal (non-helpful) wind force component (m)(WIND1) is much larger than the vertical (helpful) component (m)(WIND2). The waveform of the wind-induced acceleration normal to the missile (WINDM) is defined by Figure 4.5. The target acceleration for Sub-case 4.1.4 is described by Figure 4-6, and is a 128. ft/sec^2 (4g) evasive maneuver which drives the target upward - away from intercept .

The optimal disturbance-utilizing control for this subcase was computed as in the previous intercepts, and the resulting missile and target trajectories are shown in Figure 4-32.

The time-histories of the control force components for sub-case 4.2.4 (Figures 4-34 and 4-35) reflect the effects of winds (between t = 1.7 and t = 2.5) and target maneuvers (beginning at t = 1.1).

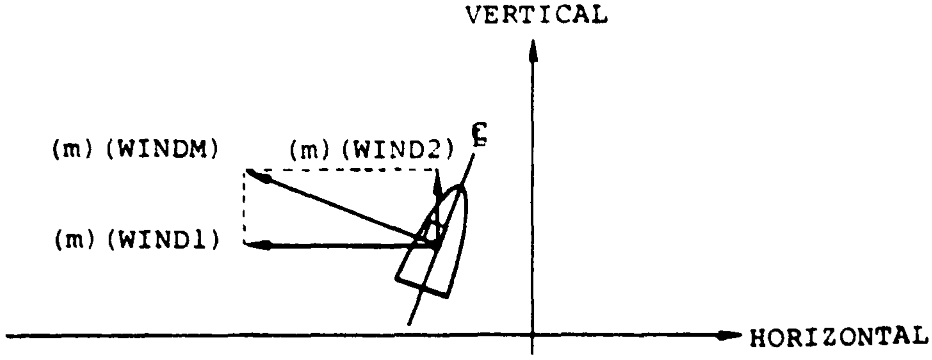

Figure 4-33. Normal wind force (m)(WINDM) and horizontal and vertical components (m)WIND1, (m)(WIND2), respectively, for Sub-case 4.2.4.

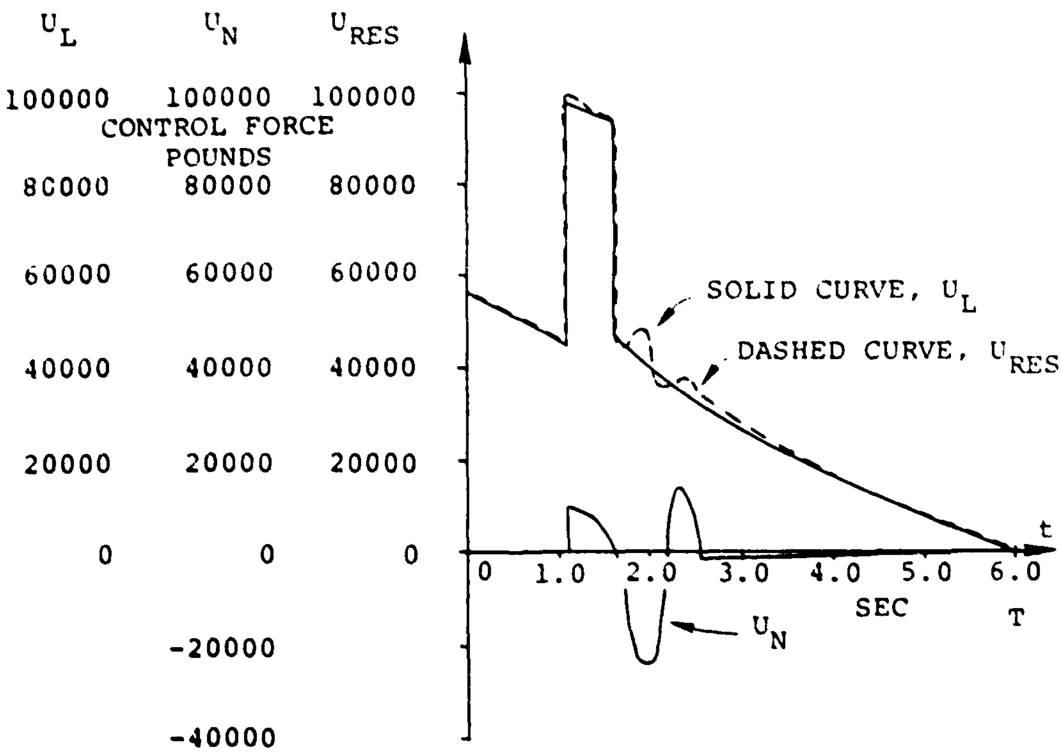

Figure 4-34. Longitudinal (U_L), Normal (U_N), and Resultant (U_{RES}) control forces for Sub-case 4.2.4; disturbance-utilizing control.

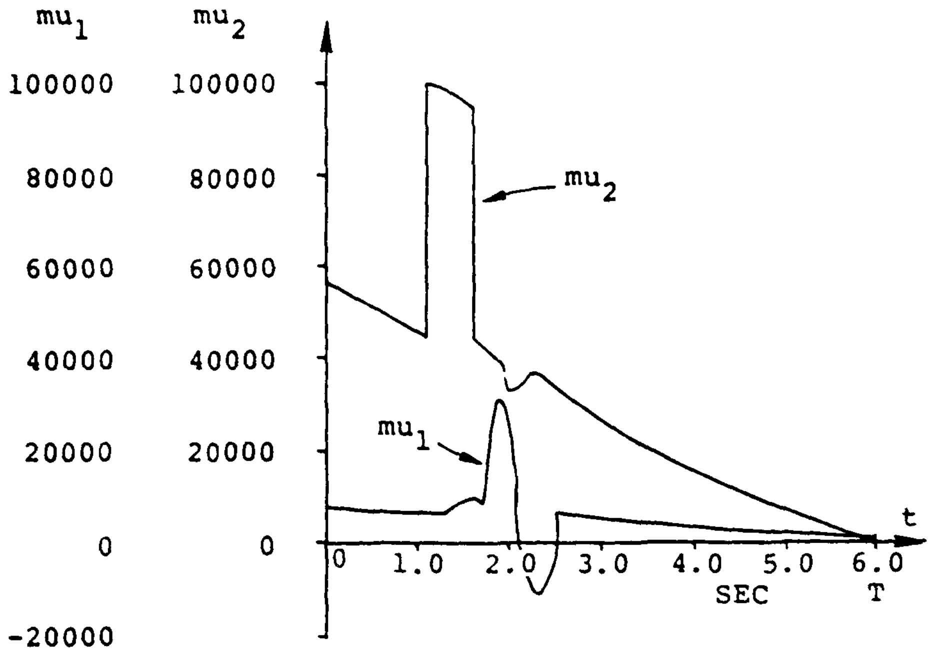

Figure 4-35. Horizontal (mu_1) and vertical (mu_2) control forces for Sub-case 4.2.4; disturbance-utilizing control.

The time-histories of the missile target states for Sub-case 4.2.4 are shown in Figures 4-36 and 4-37. The effects of the wind disturbance are seen in the horizontal position and velocity states in the interval [t = 1.7, t = 2.5]. The effect of the target's evasive maneuver, and the cumulative effect of drag on the missile, result in a slight decrease in relative vertical velocity x_4 near the final time (see Figure 4-37).

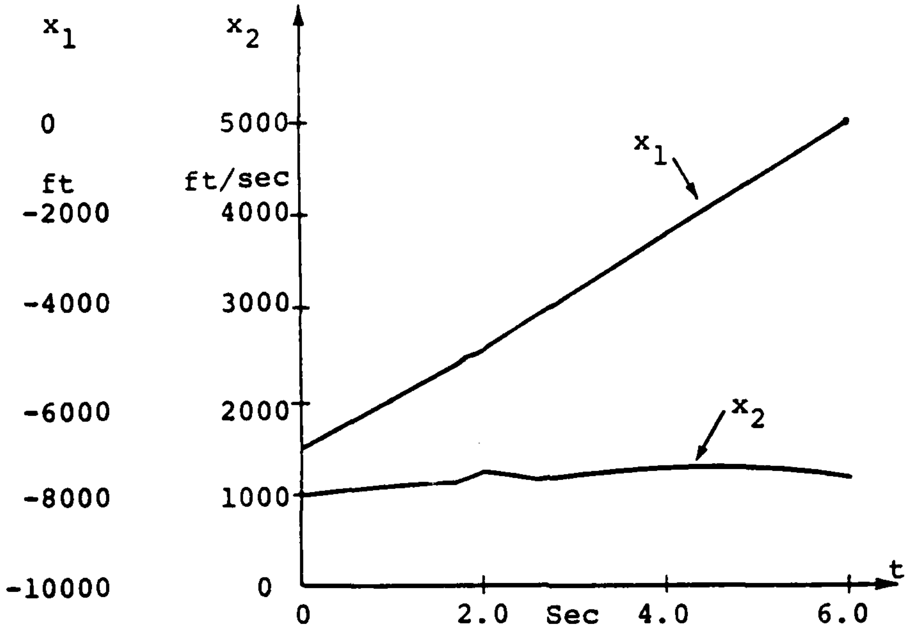

Figure 4-36. Horizontal state histories: x_1 (relative position), x_2 (relative velocity), Sub-case 4.2.4; disturbance-utilizing control.

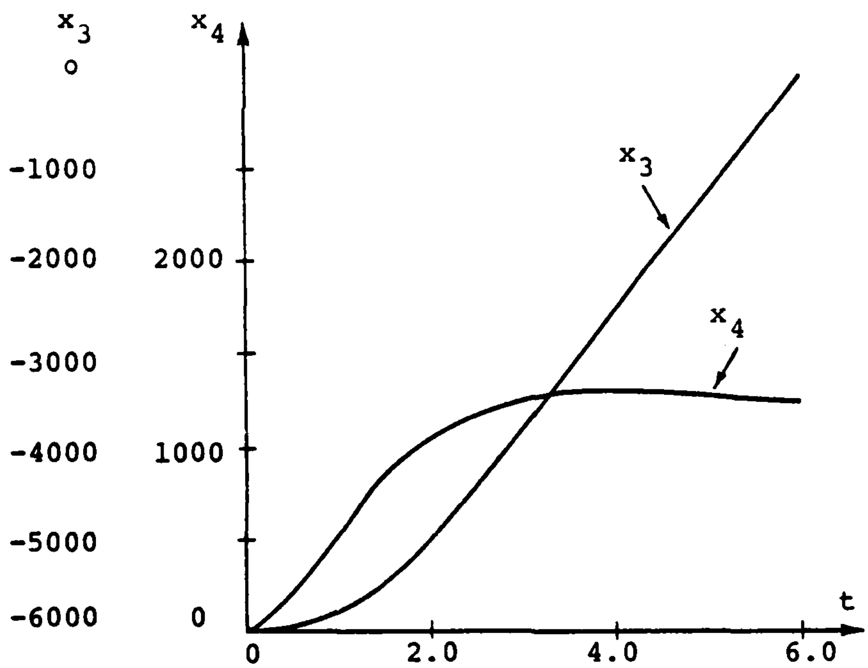

Figure 4-37. Vertical state histories: x_3 (relative position), x_4 (relative velocity), for Sub-case 4.2.4; disturbance-utilizing control.

The net disturbance effects are seen in the plots of w_1 and w_2 in Figure 4-38, which reflect the combined effects of gravity, drag, winds, and target maneuvers.

The effect of w_1 and w_2 on utility is seen in Figure 4-39. Note that $\mathscr{U}$ is negative for the total flight, showing the effect of the highly detrimental disturbance environment of Sub-case 4.2.4.

The performance of the missile with disturbance-utilizing controller for Sub-case 4.2.4 is summarized in Table 4-4. Note that the disturbance-utilizing controller achieves positive total effectiveness $\mathscr{E}_T$ even though the utility $\mathscr{U}$ is always negative for this case. This result demonstrates applications of disturbance-utilizing control are not limited to those cases in which the disturbances have positive utility $\mathscr{U}$. The disturbance-utilizing controller for this case produces significantly better performance than the LQ controller, in terms of terminal miss-distance, control energy requirements, and control fuel consumption, even though the available "utility of disturbances" $\mathscr{U}$ is never positive.

4.3 A Planar - Motion Homing Intercept Problem with Fixed Target and Disturbance-Utilizing Control

4.3.1 Mathematical model. In this section we consider a homing intercept problem in which a missile is to be controlled during the final phase of its flight so that its position coincides with that of a fixed target at

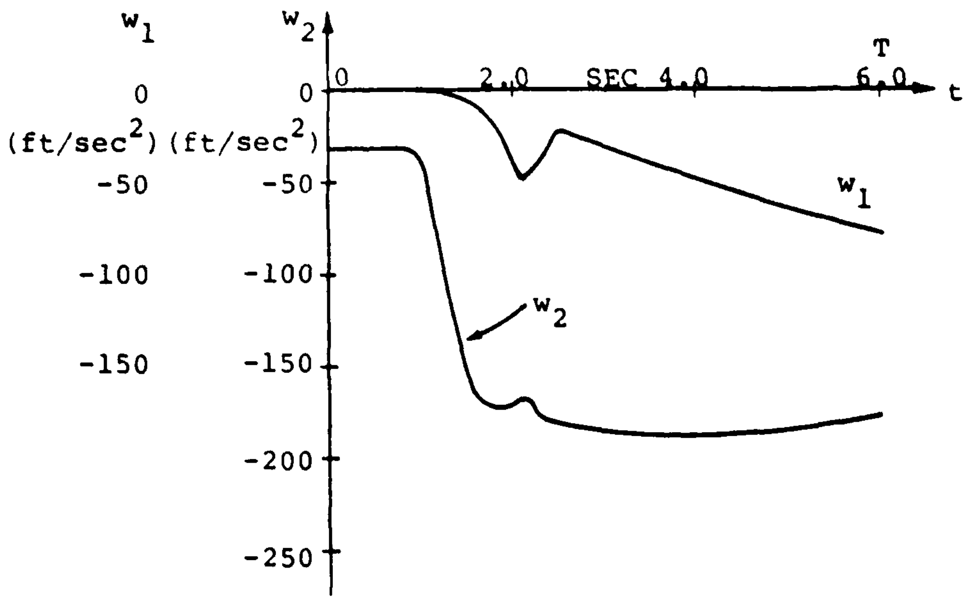

Figure 4-38. Relative accelerations due to disturbances: w_1 (horizontal) and w_2 (vertical), for Sub-case 4.2.4.

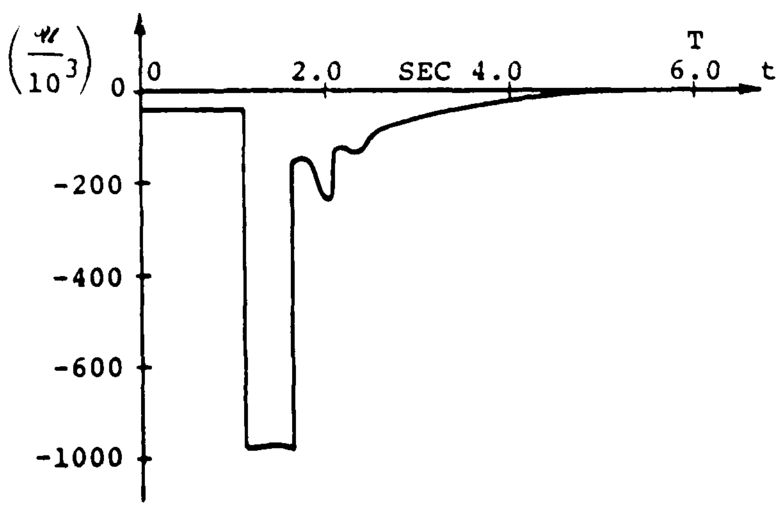

Figure 4-39. Disturbance utility for Sub-case 4.2.4.

TABLE 4-4. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.2.4.

	PERFORMANCE INDEX J	ϵ_T	CONTROL ENERGY EC	CONTROL FUEL FAU	HORIZ. MISS $x_1(T)$ (FT)	ϵ_{M_H} (%)	VERT. MISS $x_2(T)$ (FT)	ϵ_{M_V}	MISS DISTANCE MD (FT)	ϵ_{MD}
DUC	0.513×10^6	25.1	0.513×10^6	1092.0	-0.5	94.3	-0.7	96.9	0.83	96.6
LQ	0.685×10^6		0.656×10^6	1433.0	-8.0		-22.8		24.1	

NOTE: SEE PAGE 157 FOR DEFINITIONS OF J, ϵ_T, EC, FAU, ϵ_{M_H}, ϵ_{M_V}, MD and ϵ_{MD}

a specified terminal time, even in the face of disturbances which may, or may not, be detrimental to the control objective. The planar geometry for this problem is shown in Figure 4-40, where the origin is located at the fixed ground target position and the position of the missile is defined by the coordinates (X_M, Y_M), where X_M is horizontal and Y_m is vertical, relative to the ground.

It is convenient to consider a reference line-of-sight (REF LOS) passing through the target and oriented at a known angle α_h relative to the horizontal line X_M. The REF LOS is established *a priori*, and may correspond to a desired orientation of the line-of-sight. A coordinate x_1 is established normal to the REF LOS (Figure 4-40) and it is assumed that the missile begins the homing phase of the problem with a certain displacement $x_1(o)$ and velocity

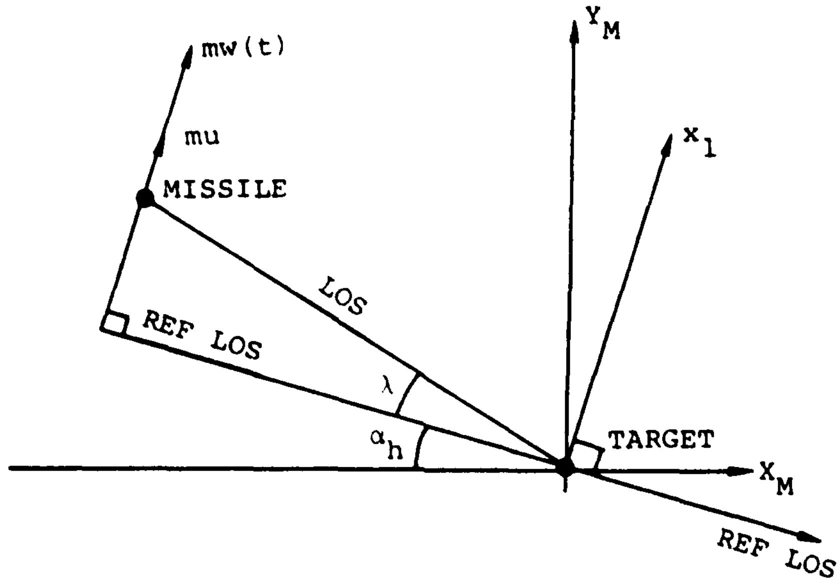

Figure 4-40. Coordinate system for small line-of-sight angle homing intercept model.

$x_2(o)$ (where $x_2 = \dot{x}_1$) normal to the REF LOS. It is assumed that a previous "midcourse" guidance phase has delivered the missile to the beginning of the homing phase at $t = t_o$; thus, non-zero values of $x_1(o)$ and $x_2(o)$ characterize the extent to which the midcourse phase has failed to enable the missile to start the homing phase under ideal conditions. The initial range to the target and the closing velocity are assumed given. The following assumptions are made in connection with this problem:

(a) The displacement x_1 (t) of the missile at any time $t \epsilon [t_o, T]$ during the homing problem is small (and hence the line-of-sight angle λ (Figure 4-40) is small), so that the forces normal to the LOS may be considered to be approximately normal to the REF LOS.

(b) The missile is controlled by a force mu (m is the mass of the missile and u is the acceleration associated with the control force) which may be considered to be normal to the REF LOS. This type of control may, for example, be associated with

(1) A missile with "side-thrusters," oriented so that the missile's longitudinal axis is approximately along the REF LOS: or

(2) An aerodynamically controlled missile (e.g., a missile controlled by the deflection of tail fins) whose longitudinal axis lies approximately along the REF LOS.

(c) The component of closing velocity (between missile and target) along the REF LOS can be considered to be constant during the homing problem.

(d) The disturbance forces of primary interest are those which are normal to the REF LOS, represented by mw(t) in Figure 4-40, where w(t) is the acceleration normal to the REF LOS resulting from disturbance forces.

Since the initial relative range and the closing velocity (constant) is given in this problem, the time t_z at which the relative range along the REF LOS will become zero is known *a priori*. This value t_z is used as the specified terminal time T at which x_1 is to be driven to zero along the coordinate normal to the REF LOS. In missile applications, T is typically obtained from radar or range-finder data in the form of time-to-go ($t_{go} = T - t$).

Any error in the measurement (or estimate) of t_{go} will result in a non-zero miss distance along the REF LOS. Errors in the knowledge of T are not considered in the present work, but have been investigated by several authors [21], [22] in relation to homing intercept problems, and current research is underway at the U. S. Army Missile Command to find improved methods for estimating t_{go}.

Assumptions (a) through (c) and the associated geometry of Figure 4-40 define a "small LOS angle" missile homing model like that which has been used by a number of workers [23] - [34] in applications including intercept and rendezvous (where displacement and velocity normal to the REF LOS are driven to small values as $t \to T$). However, the "small LOS angle" model is used in the present work in a unique way - disturbances normal to the REF LOS (note assumption (d)) are utilized in an optimal manner. Former approaches either ignored these disturbances, or modeled them as gaussian noise and used stochastic control approaches to cope with them. The application of the "small LOS angle" model to the missile homing problem where disturbances are present results in a particularly straight-forward implementation of the linear-quadratic disturbance-utilizing control theory.

The equations describing the motion of the missile normal to the REF LOS are

$$\dot{x}_1 = x_2 \tag{4.59}$$

$$\dot{x}_2 = u + w(t) \tag{4.60}$$

$$y = \left[x_1, x_2\right]^T \tag{4.61}$$

where the output vector Y has x_1 and x_2 as its elements - it is assumed that high-quality measurements of x_1 and x_2 are available from tracking data. These equations may be written in the form

$$\dot{x} = Ax + B\ u + F\ w \tag{4.62}$$

$$Y = C\ x \tag{4.63}$$

where

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.64}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.65}$$

$$F = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{4.66}$$

and

$$C = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{4.67}$$

It is assumed that the disturbance w(t) is a slowly-varying function of time which may be closely approximated by linear combinations of constant levels and linear ramps:

$$w(t) = C_1 + C_2 t \tag{4.68}$$

where C_1 and C_2 are unknown constants. The disturbance process is written in state-variable form as

$$z_1 = w \tag{4.69}$$

$$\dot{z}_1 = z_2 + \sigma_1(t) \tag{4.70}$$

$$\dot{z}_2 = 0 + \sigma_2(t) \tag{4.71}$$

or in the form

$$\dot{z} = D\, z + \sigma(t) \tag{4.72}$$

$$w = H\, z \tag{4.73}$$

where

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{4.74}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{4.75}$$

and $\sigma(t) = [\sigma_1 , \sigma_2]$ is a sparse vector-impulse sequence occurring at unknown instants.

The sources of disturbances considered in this problem are gravity and winds (when present). The gravity component acting normal to the REF LOS is

$- 32.2 \cos \alpha_h$

and a nominal value of $\alpha_h = 30$ degrees is assumed. Thus the acceleration disturbance normal to the REF LOS due to gravity is as shown in Figure 4-41.

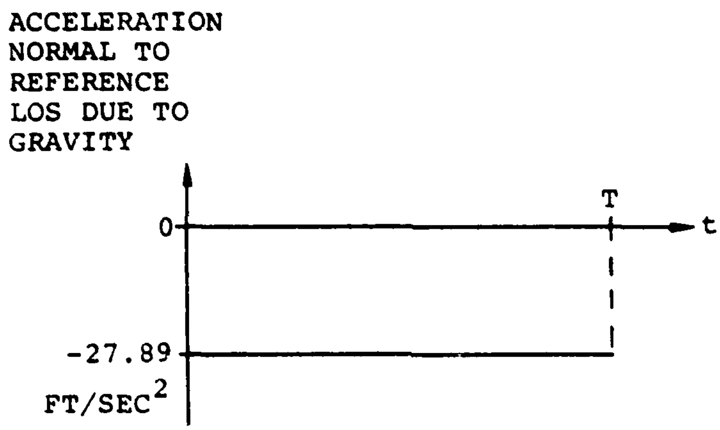

Figure 4-41. Gravity disturbance for homing intercept problem.

The acceleration disturbance due to wind in this problem is modeled by the acceleration waveform of Figure 4-5. The specific orientation of the assumed wind disturbance relative to the REF LOS will be discussed for the specific cases where wind disturbances are present.

4.3.2 Control Objective. The primary control objective for the class of problems considered in this section is to drive the displacement of the missile (normal to the REF LOS) to zero at a specified terminal time T; that is, to regulate the state x_1 to zero at $t = T$. The secondary objective is to achieve the primary objective while effectively utilizing the "free" energy of the disturbance w(t). A special case is also considered (Section 4.3.3.3) where an additional objective is to achieve a specified missile trajectory approach angle at $t = T$, while achieving the primary and secondary objectives.

The control objectives are to be achieved by minimizing the quadratic performance index

$$J = \frac{1}{2} e^T(T) S e(T) + \frac{1}{2} \int_{t_o}^{T} \left[e^T(t) Q e(t) + r u^2(t) \right] dt \qquad (4.76)$$

where $e = x_{sp} - x = - x$, subject to the plant equations (4-62) and (4-63) and the disturbance process equations (4-72) and (4-73). The terminal state weighting matrix S and the matrix Q will be numerically specified when they are used in the specific cases. The control weighting coefficient r is set to 1 for all cases to be discussed here.

4.3.3 Discussion of Results. The homing intercept problem is solved by applying the theory of Section 2.4, which leads to the composite state vector Equation (4.37), the composite system Equation (4.38) and the performance index Equation (4.39). The optimal control is computed by Equation (4.40) after computing the time-varying gains $K_x(t)$ and $K_{xz}(t)$ as the solutions of Equations (4.41) and (4.42.). The time-varying gain $K_z(t)$ is also computed (by solving Equation (4.43)) for use in computing the disturbance utility $\mathcal{U}$ for analysis purposes. The problem is solved on a CDC-6600 computer as described for the missile intercept problem in Section 4.2.3, using backward-time integration to find the initial conditions for K_x, K_{xz} and K_z.

The plant state x for the optimal control Equation (4.40) is assumed to be available from position and velocity data (as from high-quality radar tracking measurements, for example). The disturbance state vector z is assumed to be obtained from an ideal state reconstructor (estimator) for this problem; the grounds for making this assumption were discussed in Section 4.2.3 in connection with the problem of Section 4.2, and they also apply to the homing intercept problem of the present section.

4.3.3.1 Sub-case 4.3.1 - Planar Homing Intercept. Disturbance Input: Gravity (helpful). In this sub-case we consider the performance of a missile with disturbance-utilizing control in a planar homing intercept having the missile-target geometry as shown in Figure 4-42. The parameter values for Case 4.3.1 are:

a) Fixed target at 0. ft down-range, 0. ft altitude.

b) Initial missile ground-range -6778. ft

c) Initial missile altitude 4260. ft

d) Initial missile offset normal to REF LOS, $x_1(o)$ 300. ft

e) Initial missile range along REF LOS -8000. ft

f) Initial missile velocity normal to REF LOS, $X_2(o)$ 0. ft/sec

g) Missile velocity along REF LOS (constant, toward target) -2000. ft/sec

h) Angle of REF LOS from horizontal 30. deg

i) Specified terminal time T 4.0 sec

j) Disturbance: gravity, helpful

k) Control weighting parameter 1.0

l) $S = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$

m) $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

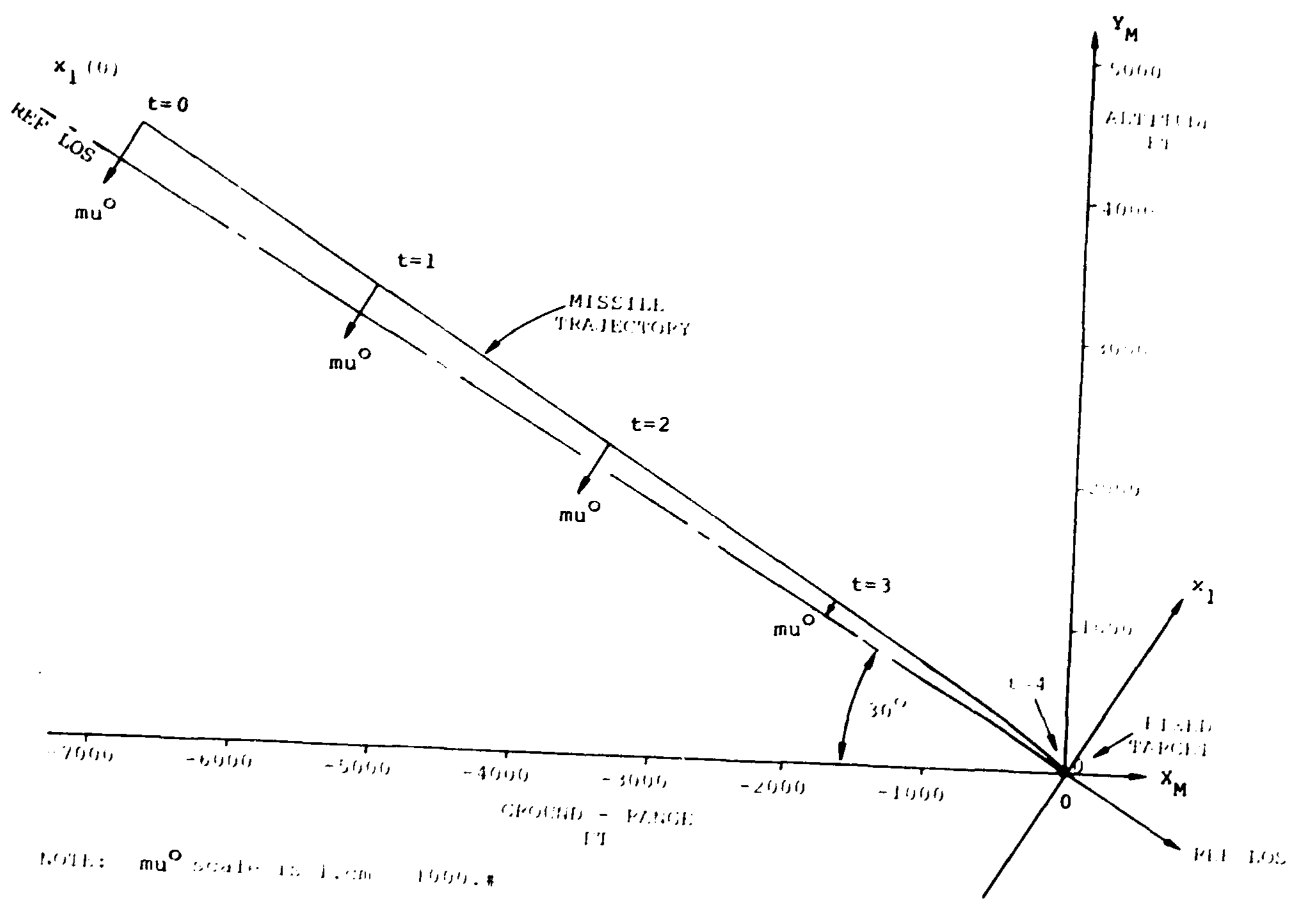

Figure 4-42. Missile trajectory for Sub-case 4.3.1, showing control force mu^o; disturbance-utilizing control. Disturbance present: gravity.

This homing-intercept problem was solved on the CDC 6600 computer, and the resulting optimally controlled missile trajectory for Sub-case 4.3.1 is shown in Figure 4-42, with the associated disturbance-utilizing control force mu^o displayed at 1 second intervals. The optimal control u^o is computed by Equation (4.40) as a function of the time-varying gain matrices K_x and K_{xz}. The time-varying missile mass m varies as shown in Figure 4-3. The missile is able to apply the control force in a direction approximately normal to the missile trajectory (assuming small angle of attack) rather than normal to the REF LOS as desired. The missile trajectory angle relative to the horizontal goes from 30 degrees at $t = o$ to 34 degrees at $t = 4.0$. Therefore, the maximum error in the angle of application of the control force is 4 degrees, which results in the application of 99.8% of the control force mu^o normal to the REF LOS. The time-history of the control force requirement is shown in Figure 4-43, which is seen to be nearly a linear function of time.

The time-histories of the states x_1 and x_2 are shown in Figure 4-44. Note that, since no penalty has been placed on $x_2(T)$, it has a relatively large value of -140 ft/sec, corresponding to the missile trajectory angle which is about 4 degrees greater than the 30 degree angle of the REF LOS. The disturbance for this case (Figure 4-45) is the projection of gravity normal to the REF LOS. The utility (Figure

4-46) is non-negative for the whole flight, as the result of the helpful action of the disturbance in this sub-case.

The performance of the missile with disturbance-utilizing control for this case is compared with that of the conventional linear-quadratic controller in Table 4-5, showing superior performance for the disturbance-utilizing controller in terms of J, $\mathscr{E}_T$, EU, EAU, $x_1(T)$ and $\mathscr{E}_{MD}$, where

$$J = \frac{1}{2} e^T(T) S e(T) + \frac{1}{2} \int_{t_o}^{T} \left[e^T(t) Q e(t) + r\, u^2(t) \right] dt \qquad (4.77)$$

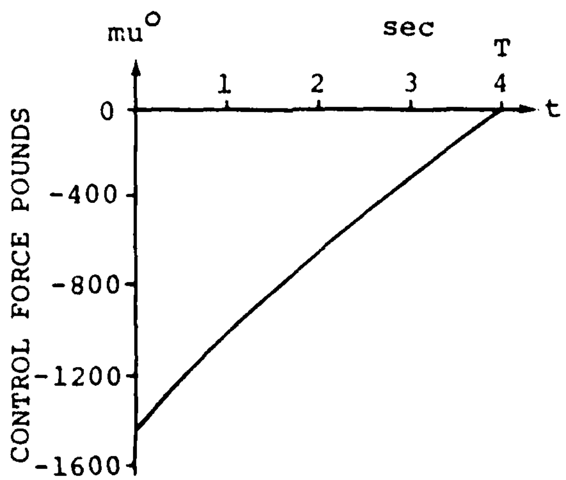

Figure 4-43. Disturbance-utilizing control force for Sub-case 4.3.1.

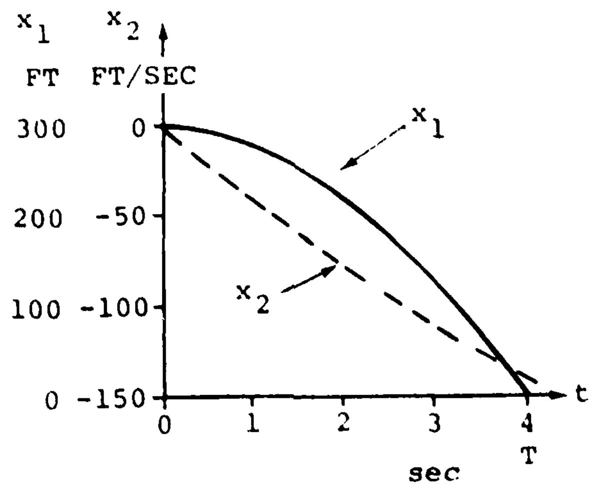

Figure 4-44. State histories: x_1 and x_2 for Sub-case 4.3.1 with disturbance-utilizing control.

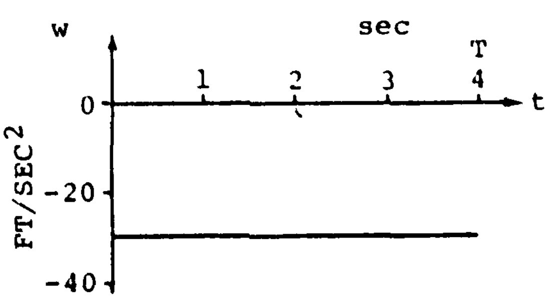

Figure 4-45. Disturbance acceleration w, for Sub-case 4.3.1.

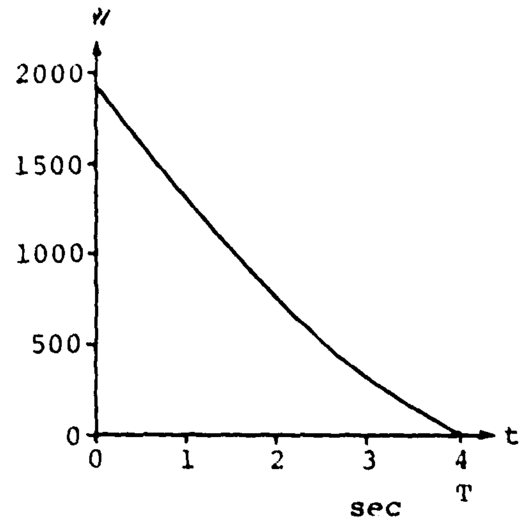

Figure 4-46. Disturbance utility for Sub-case 4.3.1.

TABLE 4-5. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.3.1.

	PERFORM-ANCE INDEX J	$\mathscr{E}_T$ %	CONTROL ENERGY EU	CONTROL FUEL EAU	MISS-DISTANCE NORMAL TO REF LOS $x_1(\bar{T})$ (FT)	$\mathscr{E}_{MD}$ %
DUC	138.0	94.9	137.0	29.0	0.4	96.6
LQ	2722.0		2047.0	113.0	-11.6	

NOTE: SEE PAGES 202 AND 205 FOR DEFINITIONS OF J, $\mathscr{E}_T$, EU, EAU AND $\mathscr{E}_{MD}$.

$$\%_T = \frac{J_{LQ} - J_{DUC}}{J_{LQ}} \times 100\% \tag{4.78}$$

$$EU = \frac{1}{2} \int_{t_o}^{T} u^2(t)\, dt \tag{4.79}$$

$$EAU = \frac{1}{2} \int_{t_o}^{T} |\, u(t)\, |\, dt \tag{4.80}$$

$$\%_{MD} = \frac{\left| x_1(T) \right|_{LQ} - \left| x_1(T) \right|_{DUC}}{\left| x_1(T) \right|_{LQ}} \times 100\% \ . \tag{4.81}$$

All effectiveness measures show a sizeable margin at T = 4.0. Values of total effectiveness $\mathscr{E}_T$ versus terminal time values are plotted in Figure 4-47, which indicates a continuing increase in $\mathscr{E}_T$ as T increases.

4.3.3.2 Sub-case 4.3.2 - Planar Homing Intercept. Disturbance Input: Gravity (non-helpful). Sub-case 4.3.2, considered in this section, examines the performance of a missile with disturbance-utilizing control in a planar homing intercept configuration where the missile-target geometry (Figure 4-48) is such that gravity is a non-helpful disturbance, and the missile's offset from the REF LOS at t = o, $x_1(o)$, is twice what it was in Case 4.3.1. The parameters for Case 4.3.2 are as follows:

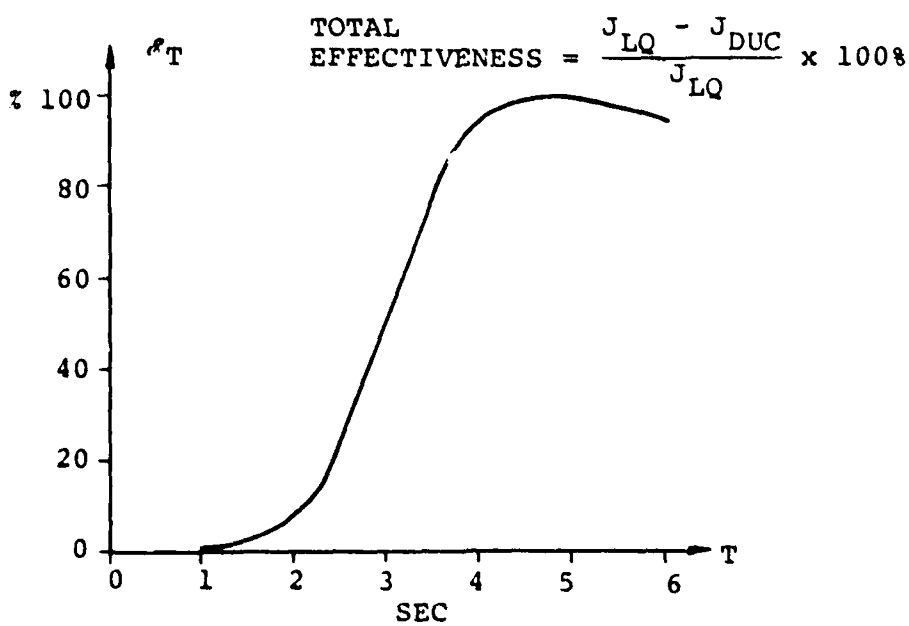

Figure 4-47. Total effectiveness $\mathscr{E}_T$ versus specified terminal time T for Sub-case 4.3.1.

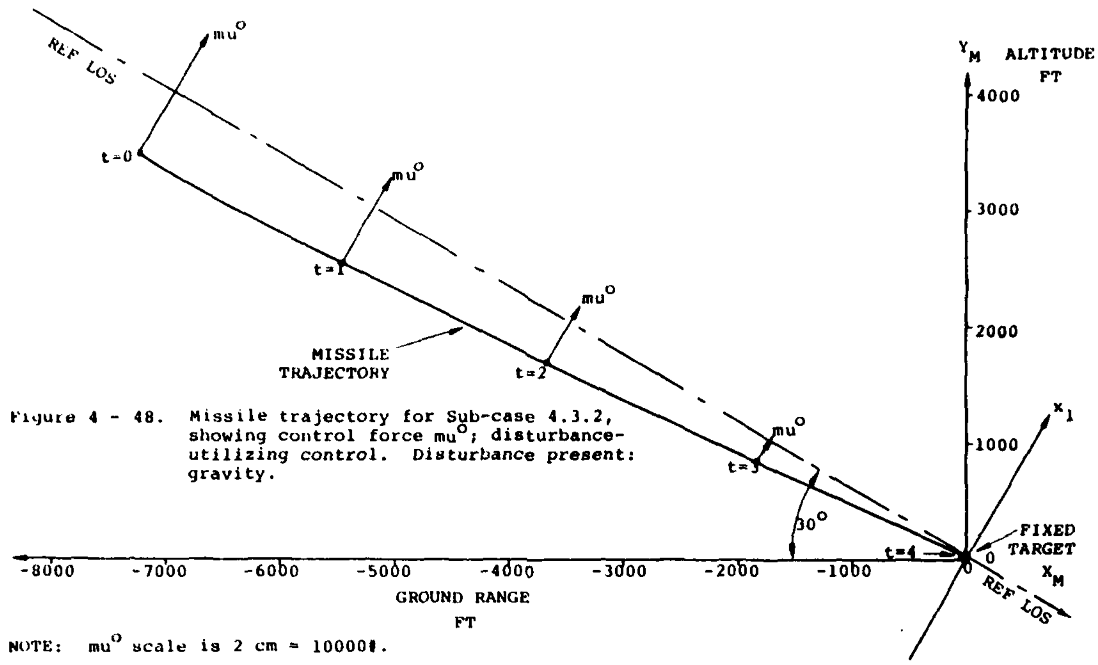

Figure 4-48. Missile trajectory for Sub-case 4.3.2, showing control force mu^{o}; disturbance-utilizing control. Disturbance present: gravity.

a)	Fixed target at 0. ft down-range, altitude.	0. ft
b)	Initial missile ground-range	-7228. ft
c)	Initial missile altitude	3480. ft
d)	Initial missile offset normal to REF LOS, $x_1(o)$	-600. ft
e)	Initial missile range along REF LOS	-8000. ft
f)	Initial missile velocity normal to REF LOS, $x_2(o)$	0. ft/sec
g)	Missile velocity along REF LOS (constant, toward target)	-2000. ft/sec
h)	Angle of REF LOS from horizontal	30. deg

i) Specified terminal time T 4.0 sec

j) Disturbance: gravity, nonhelpful

k) Control weighting parameter r 1.0

l) $S = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$

m) $Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

The computer results were obtained for Sub-case 4.3.2, and the final optimally controlled missile trajectory is shown in Figure 4-48, with the associated disturbance-utilizing control force mu^{o} displayed at 1 sec intervals. This case has a 600 ft initial offset from the REF LOS (twice that of Sub-case 4.3.1) and the geometry of this problem makes the gravity disturbance non-helpful, in contrast with Sub-case 4.3.1. As a result, the control force magnitudes for this sub-case are considerably larger

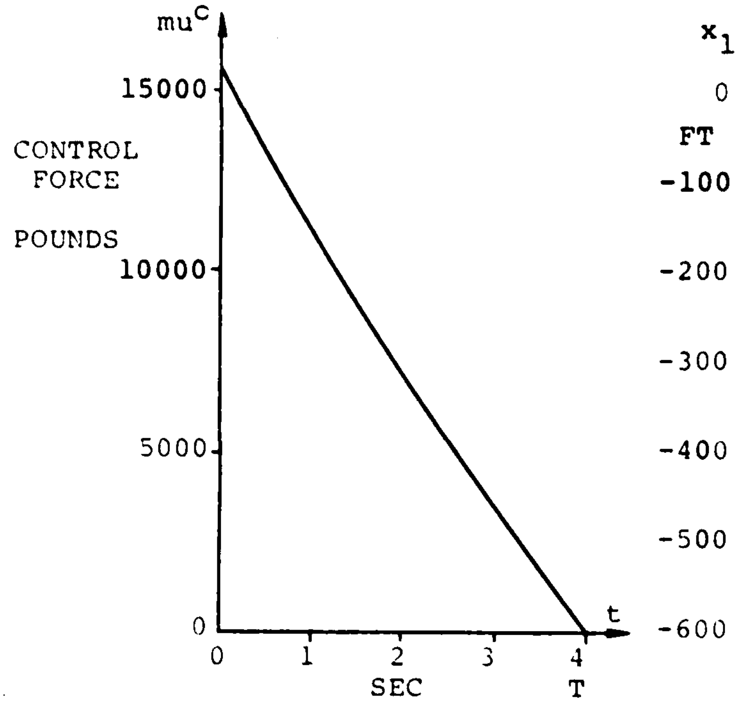

Figure 4-49. Control force for Sub-case 4.3.2; disturbance-utilizing control.

than for Sub-case 4.3.1 (see Figure 4-49). The missile trajectory angle for Case 4.2.2 goes from 30 degrees at $t = o$ to about 24 degrees at $t \rightarrow T$; the maximum error in the angle of application of the control force is -6 degrees, which results in the application of 99.5% of the control force mu^o normal to the REF LOS. As in Sub-case 4.3.1, the control force for this case (Figure 4.49) is almost a linear function of time.

The time-histories of the states x_1 and x_2 are plotted in Figure 4-50. As in Sub-case 4.3.1, no terminal

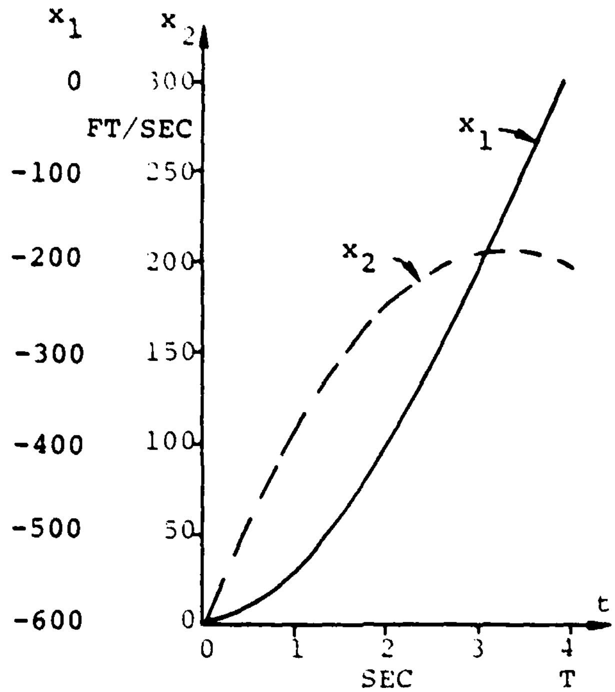

Figure 4-50. State histories: x_1 and x_2 for Sub-case 4.3.2; disturbance-utilizing control.

penalty is placed on x_2, and a relatively large value of $x_2(T)$ results. The disturbance in this case (Figure 4-51), which is the projection of the gravity acceleration normal to the REF LOS, is non-helpful, since it acts to hinder the missile from the intercept objective. As a result, the disturbance utility (Figure 4-52) is either negative or zero for the whole flight.

The disturbance-utilizing controller for Sub-case 4.3.2 performs better than the conventional linear-quadratic controller (see Table 4-6) even in the face of the totally detrimental disturbance, which indicates that, even though positive utility is <u>never</u> available, the disturbance-utilizing control law still does better in managing the states of the plant relative to the disturbance states.

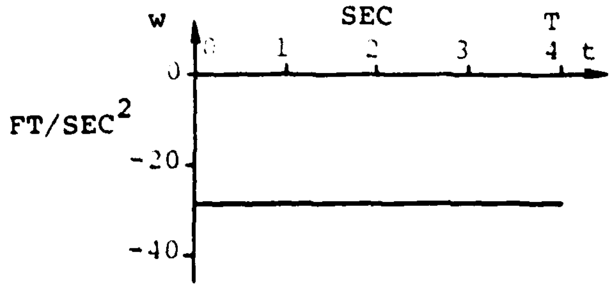

Figure 4-51. Disturbance acceleration w for Sub-case 4.3.2.

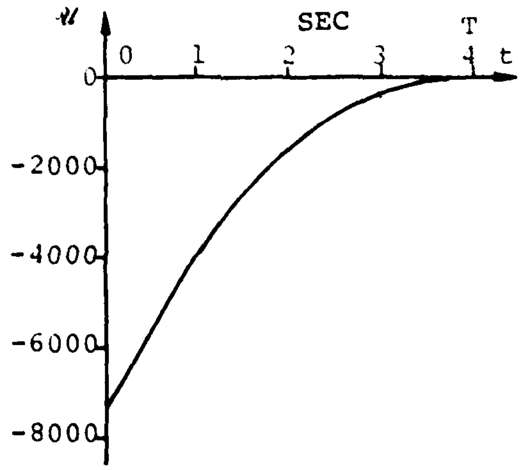

Figure 4-52. Disturbance utility for Sub-case 4.3.2.

TABLE 4-6. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.3.2.

	PERFORM-ANCE INDEX J	$\mathscr{E}_T$ %	CONTROL ENERGY EU	CONTROL FUEL EAU	MISS-DISTANCE NORMAL TO REF LOS x_1 (T) (FT)	$\mathscr{E}_{MD}$ %
DUC	0.158 X10^5	14.2	0.157 X10^5	306.0	-3.8	75.8
LQ	0.183 X10^5		0.171 X10^5	358.0	-15.7	

NOTE: SEE PAGES 202 AND 205 FOR DEFINITIONS OF J, $\mathscr{E}_T$, EU, EAU AND $\mathscr{E}_{MD}$.

The effectiveness (Figure 4-53) for Sub-case 4.3.2 shows that the disturbance-accommodating controller continues to achieve a lower J as the specified terminal time is increased.

4.3.3.3 Sub-case 4.3.3 - Planar Homing Intercept with Trajectory Angle Specification at Terminal Time. Disturbance Inputs:

a) Gravity (helpful)

b) Wind (non-helpful)

In this sub-case we consider the performance of a missile with disturbance-utilizing control in a planar homing problem which has the primary objective of intercepting the target and secondary objectives of

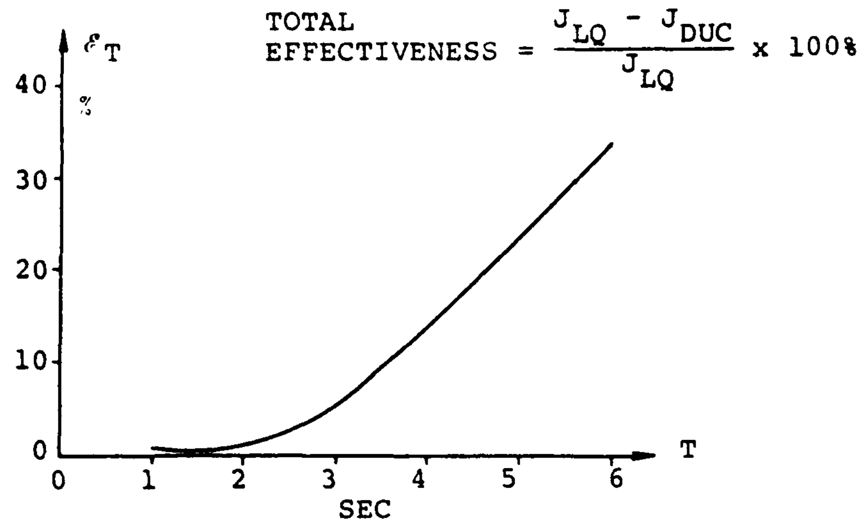

Figure 4-53. Total effectiveness $\mathscr{E}_T$ versus specified terminal time T for Sub-case 4.3.2.

(1) Achieving a specified orientation of the velocity vector angle (the trajectory angle) at terminal time, and

(2) Effectively utilizing "free" disturbance energy.

Realization of the angle specification leads to a consideration of a terminal weighting matrix S with non-zero weighting on $x_2(T)$. The missile-target geometry for this sub-case is shown in Figure 4-54. The parameter values for Sub-case 4.3.3 are:

a) Fixed target at 0. ft down-range, 0. ft altitude.

b) Initial missile ground-range -6778. ft

c) Initial missile altitude 4260. ft

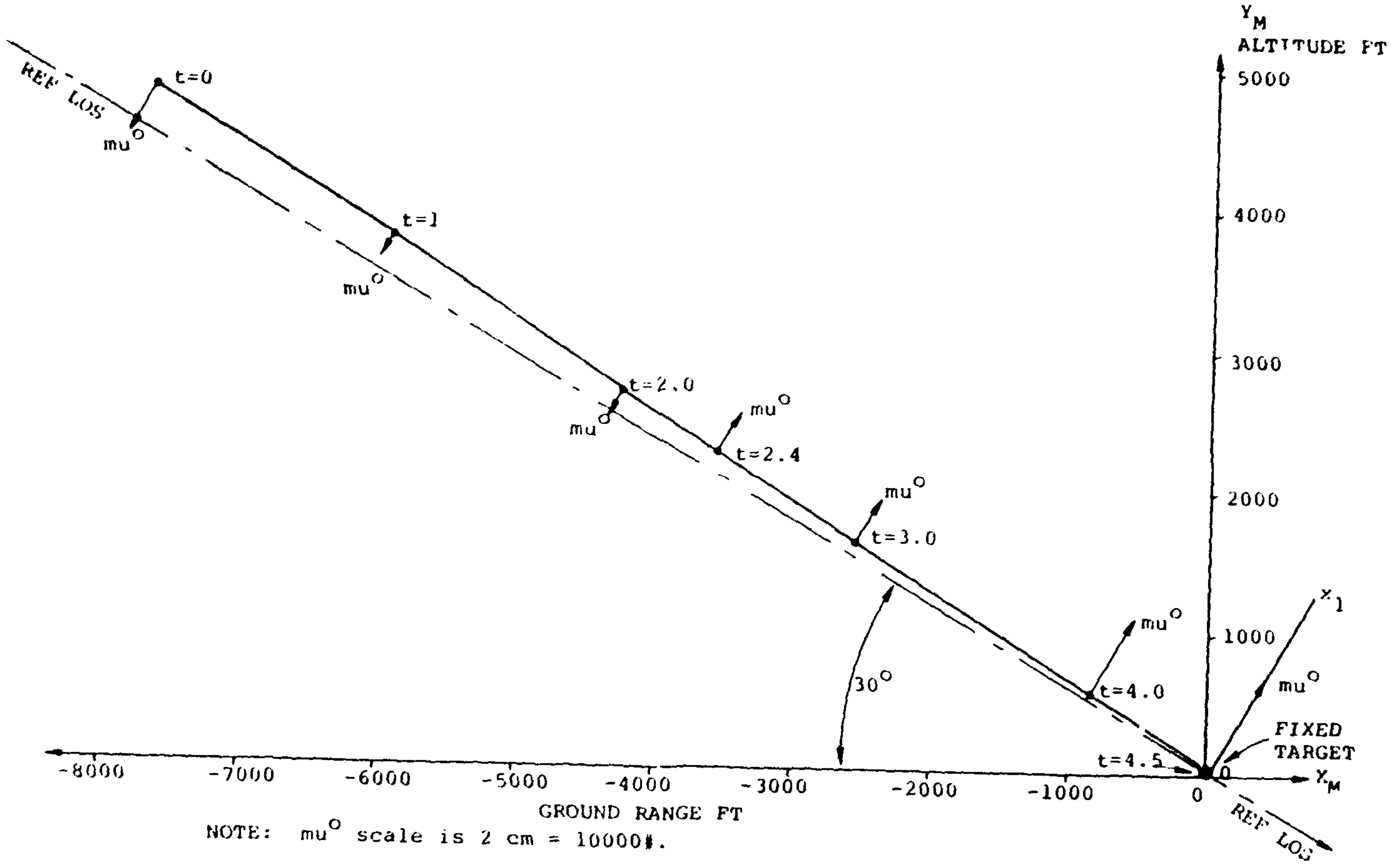

Figure 4-54. Missile trajectory for Sub-case 4.3.3, showing control force mu°; disturbance-utilizing control. Disturbances present: a) gravity b) winds

d) Initial missile offset normal to REF LOS, $x_1(o)$ — 300. ft

e) Initial missile range along REF LOS — -9000. ft

f) Initial missile velocity normal to REF LOS, $x_2(o)$ — 0. ft/sec

g) Missile velocity along REF LOS (constant, toward target) — -2000. ft/sec

h) Angle of REF LOS from horizontal — 30. deg

i) Specified terminal time T — 4.5 sec

j) Disturbances: gravity (helpful) and wind (nonhelpful)

k) Control weighting parameter r — 1.0

l)

$$S = \begin{bmatrix} 50 & 0 \\ 0 & 10 \end{bmatrix}$$

to achieve weighting on $x_1(T)$ <u>and</u> $x_2(T)$.

m)

$$Q = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

The optimal disturbance-utilizing control for sub-case 4.3.3 was computed as in the previous homing intercept problems, and the resulting missile trajectory is shown in <u>Figure 4-54</u>, with the disturbance-accommodating control forces displayed at selected times. Note the increased level of mu_o at t = 2.0 (compared with the level at t = 1.0) required to handle the wind disturbance which begins at t = 1.7 seconds and ends at t = 2.5. The control at t = 2.5

reflects the wind disturbance and the requirement to achieve the specified angle at $t = T$. The result is a reversal in the sign of mu^o for the remainder of the flight. Considerable control force levels are required for this case as $t \to T$, as distinguished from the previous cases (with no angle specification) which had the control approaching zero for $t \to T$.

The effects of the wind and the angle specification on the control are clearly seen in <u>Figure 4-55</u>, where a large transient is required between $t = 1.7$ and $t = 2.5$ to handle the wind and the control force continues to increase as $t \to T$. No wind disturbance effects are seen in the behavior of the states x_1 and x_2 (<u>Figure 4-56</u>), indicating that the disturbance-utilizing control is doing very well in accommodating the disturbance. The effect of the velocity weighting term in S is seen as the decrease in the value of x_2 as $t \to T$. This assures that the velocity vector of the missile lies almost parallel to the reference LOS as $t \to T$; i. e., along a direction very close to 30 degrees from the horizontal. To achieve an <u>arbitrary</u> angle α_h relative to horizontal, it is only necessary to specify the value of α_h at the beginning of the problem, which automatically defines the x_1 and reference LOS coordinates. This feature of the model, which was suggested in [35], is applicable to a general class of homing intercept problems where the approach angle is important.

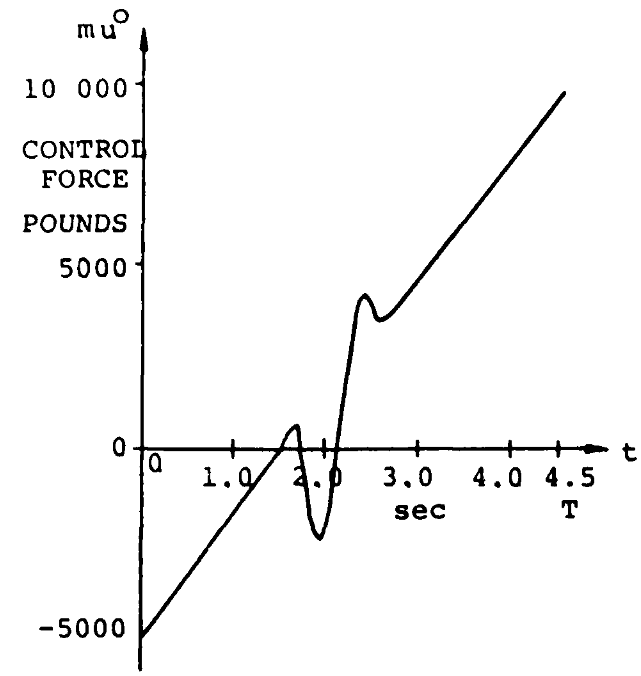

Figure 4-55. Control force for Sub-case 4.3.3, disturbance-utilizing control.

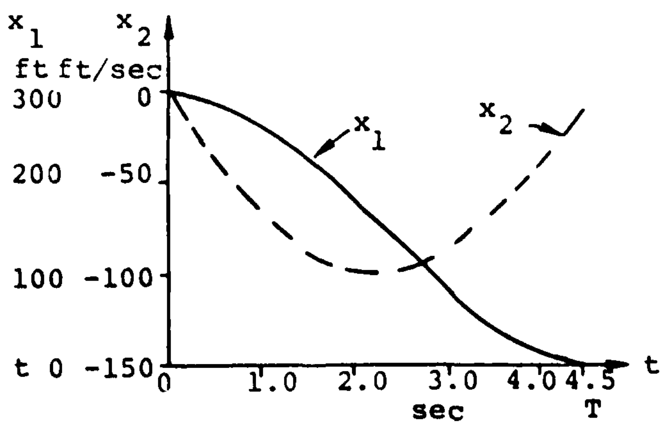

Figure 4-56. State histories: x_1 and x_2 for Sub-case 4.3.3, disturbance-utilizing control.

Figure 4-57 shows the disturbance inputs for this problem. The wind disturbance waveform is as given in Figure 4-5; the orientation of the wind for Sub-case 4.3.3 is along the positive x_1 coordinate, such that the wind acceleration WINDM of Figure 4-5 is tending to move the missile away from the REF LOS. The gravity component along x_1 is negative, acting to move the missile toward the REF LOS.

The effects of the disturbances result in a utility value which is almost always negative (Figure 4-58) except for two very brief positive excursions resulting from the derivative of the disturbance. Recall that the utility function (Equation 2.108) depends on all elements of the disturbance vector z, which includes $z_1 = w$ and $z_2 = \dot{z}_1 = \dot{w}$ in the disturbance model (Equations (4-69) - (4-73)) being used).

The performance of the missile with disturbance-utilizing control is summarized in Table 4.7, where it is compared with that of the conventional linear-quadratic controller. The disturbance-utilizing controller achieves a lower value of J (and therefore a positive total effectiveness $\mathscr{E}_T$); uses less control energy, based on EU; uses less fuel, based on EAU; and achieves a lower value of $x_1(T)$. Inclusion of terminal velocity weighting in this problem results in trajectory angles (at $t = T$) of -0.307 degrees and -1.52 degrees (relative to the 30 degree orientation of the REF LOS) for the disturbance-utilizing

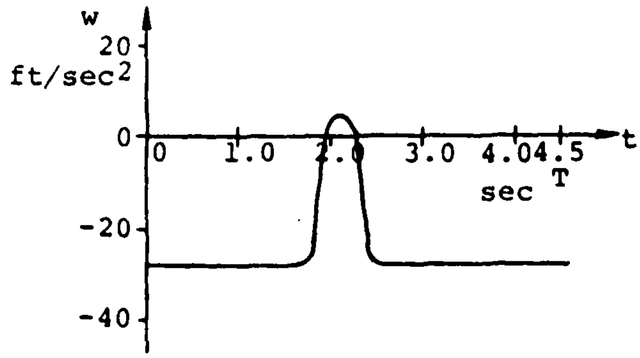

Figure 4-57. Disturbance acceleration w_1 for Sub-case 4.3.3.

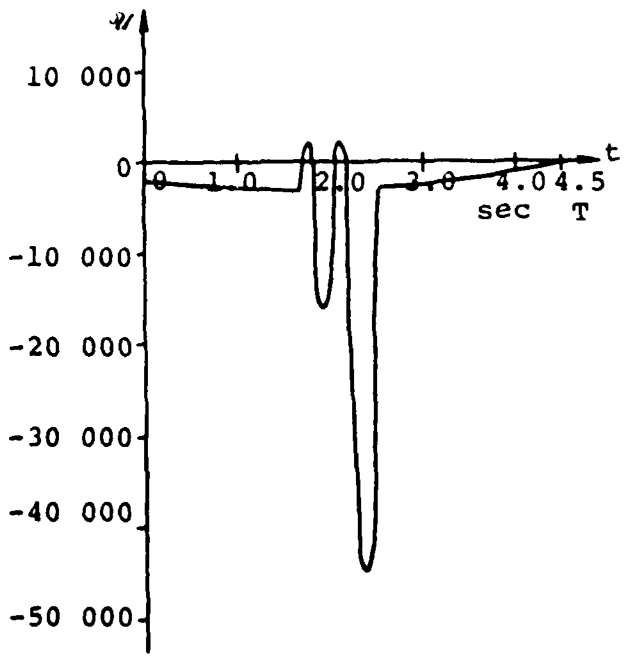

Figure 4-58. Disturbance utility for Sub-case 4.3.3.

TABLE 4-7. PERFORMANCE OF DISTURBANCE-UTILIZING CONTROLLER COMPARED WITH CONVENTIONAL LINEAR-QUADRATIC CONTROLLER FOR SUB-CASE 4.3.3.

	PERFORM-ANCE INDEX J	ε_T	CONTROL ENERGY EU	CONTROL FUEL EAU	MISS-DISTANCE NORMAL TO REF LOS x_1 (T) (FT)	ε_{MD} %	TRAJECTORY ANGLE ERROR AT t=T (DEGREES)	ε_A %
DUC	0.711×10^4	13.5	0.652×10^4	200.0	0.7	85.1	-0.307	-100
LQ	0.821×10^4		0.753×10^4	229.0	-4.7		-0.152	

NOTE: SEE PAGES 202, 205, AND 220 J, ε_T, EU, EAU, ε_{MD} AND ε_A.

and conventional LQ controllers, respectively. This is reflected in the large <u>negative</u> value of ε_A, where

$$\varepsilon_A = \frac{(\text{angle error at } t=T)_{LQ} - (\text{angle error at } t=T)_{DUC}}{(\text{angle error at } t=T)_{LQ}} \times 100\% \qquad (4\text{-}82)$$

However, both controllers show very good control of the trajectory angle at t = T. Note that the DUC in this sub-case achieves much smaller miss-distance than the LQ controller, at the expense of a larger angle error at t = T. In those applications where larger trajectory angle errors can be tolerated, less weight may be placed on the velocity state (in matrix S), which should lead to lower values of control energy and fuel comsumption, and/or lower values of miss-distance $x_1(T)$. The total effectiveness $\mathcal{E}_T$ for this sub-case is positive for various values of specified terminal time T (see Figure 4-59), demonstrating that the DUC performance is consistently better than that of the LQ controller.

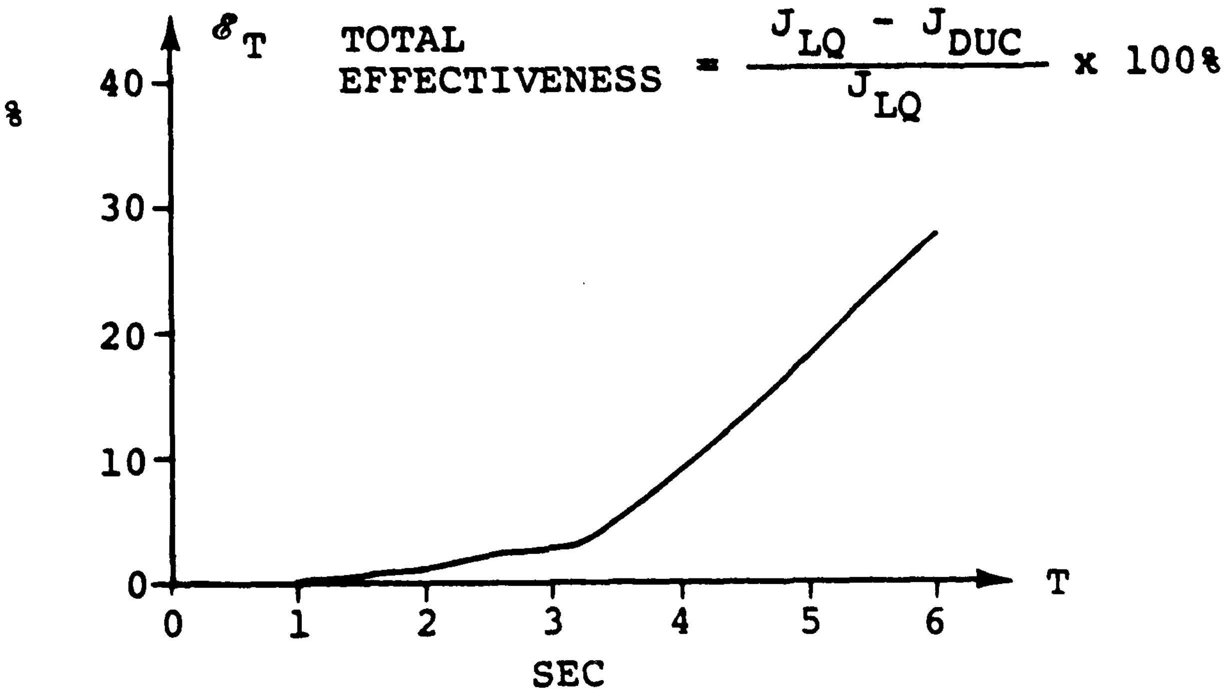

Figure 4-59. Total effectiveness $\mathcal{E}_T$ versus specified terminal time T for Sub-case 4.3.3.

CHAPTER V

CONCLUSIONS AND RECOMMENDATIONS FOR FURTHER WORK

5.1 Introduction

The research described in this document constitutes the first application of disturbance-utilizing control theory to missile guidance problems. In fact, the work described herein is apparently the most substantial application of disturbance-utilizing control theory so far attempted for any control problem [38]. This chapter presents the conclusions of this research and offers recommendations for further work.

5.2 Conclusions

The results of this investigation have demonstrated that, in many cases, the disturbance-utilizing controller produces significantly better performance than the conventional linear-quadratic controller. This superior performance is realized even when the relationship between the plant state x and the disturbance state z is such that the disturbance utility $\mathcal{U}$ is never positive during the control interval. The concept of "effectiveness" and associated effectiveness measures such as total effectiveness

$\mathcal{E}_T$, and miss-distance effectiveness $\mathcal{E}_{MD}$ has been introduced as a means of relating practical aspects of the performance of the disturbance-utilizing controller to those of a conventional linear-quadratic controller under the same conditions. By means of these performance measures it has been shown that a large class of missile/target/disturbance scenarios exists for which the disturbance-utilizing controller provides significant improvements over the conventional LQ controller in terms of practical criteria used in the missile industry. For example, in the seven missile guidance cases considered in Chapter IV, the terminal miss-distance obtained by the disturbance-utilizing controller ranged from 3.4% to 56.1% of the terminal miss-distance obtained by a conventional linear-quadratic controller under identical conditions. In addition, although the seven missile guidance cases of Chapter IV were designed with minimum miss-distance as the primary control objective, in most cases the disturbance-utilizing controller required less control energy and less control fuel than a conventional linear-quadratic controller. In one homing intercept problem (Sub-case 4.3.1) the control energy and control fuel required by the disturbance-utilizing controller were only 6.7% and 25.7%, respectively, of the control energy and control fuel required by a conventional linear-quadratic controller.

The necessary and sufficient conditions have been found for the existence of steady-state solutions of the six unilaterally-coupled matric differential equations which govern the solution of the set-point and servo-tracking problems in disturbance-utilizing control. These conditions determine the existence of steady-state disturbance-utilizing control laws. Futhermore, it has been shown that these steady-state solutions, when they exist, are solutions of certain matric algebraic equations. Numerical computational approaches have been suggested for obtaining the steady-state solutions.

Several mathematical/geometric properties of the utility function $\mathcal{U}$, in the case of time-invariant systems, have been found; namely,

(a) An expression has been obtained for the zero-utility boundary, in the set-point regulator/servo-tracking disturbance-utilizing problem, for the case of $\rho > n + \nu$ (ρ is the dimension of the disturbance state vector z, n is the dimension of the plant state vector x, and ν is the dimension of the set-point vector c).

(b) The size of the positive-utility domain in the set-point regulator/servo-tracking disturbance-utilizing problem with $\rho = n = \nu = 1$ is described through the introduction of a new parameter θ_{cz} (the angle, in the c-z plane, between the zero-utility planes). This parameter is used in conjunction with the previously-used paramenter θ_{xz} (the angle, in the x-z plane, between the zero-utility planes) to allow a graphical interpretation of the size of the positive-utility domain throughout the control interval.

(c) The limiting behavior of the utility function $\mathcal{U}(t)$ and its derivative $d\mathcal{U}/dt$ has been determined for $t \to T$ (i.e, at the terminal time) and for the steady-state condition (where "backward-time" $\tau \to \infty$). In addition, the critical-point condition which characterizes the location of maximum $\mathcal{U}$ with respect to z in (x, c, z)-space has been derived.

In this document, derivations have been presented for the steady-state gain expressions associated with

(1) a scalar regulator with a constant disturbance, for both the zero set-point and non-zero set-point cases;

(2) a scalar regulator with an exponentially-decaying disturbance, for both the zero set-point and non-zero set-point cases; and

(3) the zero set-point regulator with a second-order plant and a vector (two-dimensional) disturbance.

In addition, numerical solutions have been obtained for each case, which allow the performance of the disturbance-utilizing controller to be compared with that of a conventional linear-quadratic controller.

The robustness of the disturbance-utilizing control law relative to a mismatch between the disturbance model and actual disturbance inputs has been examined in this study. Several missile guidance cases are considered in which disturbance inputs (e.g., aerodynamic drag and winds) are <u>approximately</u> modeled by the assumed mathematical disturbance process. Specific cases of intercept problems have indicated that the match between drag waveforms and the assumed "step plus-ramp" second-order disturbance model is excellent. Some performance degradation can be detected when the faster-changing wind waveforms are experienced, but the overall effectiveness remains high as long as the faster modes of the waveforms do not constitute a dominant part of the disturbance. Whether or not one should attempt to modify the mathematical disturbance model to obtain a closer match to real-life disturbance waveforms depends on the results of a trade-off between increased design complexity and incremental performance improvement to be gained.

A unique digital-computer analysis tool (DUCAT -- Disturbance-Utilizing Control Analysis Technique) has been developed for implementing the disturbance-utilizing control law, the equations of the plant being controlled, and the disturbance models. The computer program also implements the corresponding conventional linear-quadratic control law for comparative analysis in terms of the effectiveness of the disturbance-utilizing control law. A key feature of DUCAT is the capability of obtaining the "T-Minimin" value of the optimized performance index $J^o[T]$ for a given disturbance utilizing or conventional linear-quadratic problem. The program determines the optimal values $J^o[T_i]$ for a selected set of values of specified terminal times T_i in some specified interval $[T_{min}, T_{max}]$. Then, the program selects the minimum optimal value J^o_{min} among that set and then displays the optimal control and trajectory for the particular T_{min} corresponding to J^{o}_{min}.

The control of an air defense interceptor missile in the face of a disturbance environment consisting of gravity, aerodynamic drag, winds, and target maneuvers has been found in the context of a general planar geometry configuration. In this approach, the components of the disturbance-utilizing control are determined along the missile's longitudinal and lateral coordinates. The problem of controlling a homing missile to a ground-based target in the face of gravity, winds, and a specified trajectory approach angle has been solved by using a so-called "small line-of-sight angle" geometry, in which the disturbance-utilizing control force is found as a force normal to a reference line-of-sight passing through the target position. Although the "small line-of-sight angle" homing model as applied herein is used for fixed-target cases, this model may also be applied to the disturbance-utilizing control of a missile in an air defense role against a maneuvering target. The disturbance-utilizing controller is seen to be well suited to accommodating typical target maneuver waveforms.

5.3 Recommendations for Further Work

The present study has uncovered several areas for further work in disturbance-utilizing control theory. In particular, it is suggested that further work be directed as follows:

(a) The application of disturbance-utilizing control should be considered for a three-dimensional, 6-degree-of-freedom missile intercept model with complex aerodynamics, control limits, autopilot dynamics, and a non-ideal state-reconstructor. This would be a logical follow-on to the present work. Computational algorithms for obtaining the time-varying gain matrices $K_x(t)$ and $K_{xz}(t)$ should also be investigated as part of this applications-oriented task.

(b) The application of disturbance-utilizing control to a discrete missile control problem would be useful, in view of the trend toward using sampled-data and microprocessor techniques in future missile designs. Relevant work in this area includes investigating ways to obtain the time-varying gain matrices for this problem via solutions of difference equations or by implementation of alternative, discrete algorithms for generating these matrices. Other work may include the formulation of the discrete plant/disturbance state reconstructor (estimator) for the discrete missile control problem.

(c) The design of the performance index for disturbance-utilizing control appears to be a fruitful area for further study. One question of particular interest is how to choose the performance index parameters S, Q, and R, in the general case, to maximize utility and effectiveness. Also, how to enhance utility and effectiveness (possibly by using time-varying parameters Q and R) during critical parts of the control interval, such as, for example, near the terminal time. Another question of interest is how to choose S, Q, and R in a systematic way to achieve missile trajectory shaping and to reduce the sensitivity of intercept performance to errors in estimating time-to-go. In this regard, results from the homing intercept Case 4.3.3 of the present work indicate that the use of non-zero weighting on the "velocity" state at terminal time (in the S matrix) produces terminal performance which is less sensitive to the choice of specified terminal time T than for the standard intercept problem where only the position states are weighted.

(d) An investigation of multi-mode disturbance-accommodation in the missile guidance problem should be profitable. For example, some missile intercept scenarios may call for a combination of disturbance absorption/ minimization and disturbance-utilization. An interesting question in this regard is how to relate mode-switching criteria to utilization and effectiveness levels, recognizing that positive <u>effectiveness</u> may be obtained, in the disturbance-utilizing mode, even when the <u>utility</u> is negative. The design of disturbance-absorbing controllers by the generalized "algebraic/stabilization method" [39] may be useful in such problems.

(e) A systematic method for finding a dynamic mathematical model corresponding to actual measurements of real-life disturbance waveforms would be especially useful in applications such as missile intercept, where the composite disturbance waveform may be quite complex. A related useful area of study would be the investigation of a systematic way to choose a disturbance model with a specified level of approximation error relative to an expected disturbance waveform.

(f) Many potential applications of disturbance-utilizing control are suggested, in both regulator and servo-tracking problems, by the critical need to conserve energy and fuel. The efficiency of the disturbance-utilizing approach is readily seen in the cases considered in the present study, where significant savings of control energy and control fuel consumption are realized. Other candidate applications for appreciable savings in control energy and fuel appear to be automobile speed and steering control, aircraft flight path control and fully automatic landing systems, ship steering systems, pointing and tracking systems for large antenna arrays and spacecraft control in the presence of disturbances such as gravitational fields.

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APPENDIX A

AN EXAMPLE DESIGN OF A REDUCED-ORDER COMPOSITE-STATE RECONSTRUCTOR (ESTIMATOR)

A.1 Introduction.

This Appendix describes an example design of a reduced-order composite-state reconstructor (estimator) for a system with a second-order plant and a second-order disturbance model. The purpose of this Appendix is to illustrate the design approach to, and final form of, a composite-state reconstructor for a typical missile application. The plant and disturbance models used in this example correspond to the missile homing problem considered in Section 4.3 of Chapter 4, with the exception that in the present case it is assumed that the output y consists of only the one state element x_1. The only on-line input data available to the state reconstructor is the output y(t) and the control u(t).

Johnson [5], [36] has formulated and solved the problem of observing the states x and z of the general time-varying linear dynamic system

$$\dot{x} = A(t)\, x + B(t)\, u(t) + F(t)\, w(t) \qquad \text{(A-1)}$$

$$Y = C(t)\ x \tag{A-2}$$

$$\dot{z} = D(t)\ z + M(t)\ x + \sigma\ (t) \tag{A-3}$$

$$w(t) = H(t)\ z + L(t)\ x \tag{A-4}$$

where x is an n x 1 plant-state vector, A is an n x n plant matrix, B is an n x r input matrix, u is an r x 1 control vector, F is an n x ρ disturbance input matrix, w is a p x 1 disturbance vector, y is an m x 1 output vector, z is a ρ x 1 disturbance state vector, D is a ρ x ρ disturbance process matrix, M is a ρ x n matrix, σ is a sparse sequence of impulses, H is a ρ x ρ disturbance output matrix and L is a ρ x n matrix. One "recipe" for building a physically realizable device which operates on the output y and the control u to produce estimates of x and z is given by [5], [38]

$$\begin{pmatrix} \dot{\hat{x}} \\ \hline \dot{\hat{z}} \end{pmatrix} = \left[\begin{array}{c|c} A + FL + K_1C & FH \\ \hline M + K_2C & D \end{array}\right] \begin{pmatrix} \hat{x} \\ \hline \hat{z} \end{pmatrix} - \left[\begin{array}{c} K_1 \\ \hline K_2 \end{array}\right] y(t) + \left[\begin{array}{c} B \\ \hline 0 \end{array}\right] u(t) \tag{A-5}$$

where all the matrices may be time-varying, $\hat{x}$ and $\hat{z}$ are estimates of the plant and disturbance states, and K_1 and K_2 are gain matrices which are chosen to stabilize the solution of Equation A-5. The state reconstructor Equation A-5 has dimension (n + ρ). In a 1971 paper [6] Johnson derived a different form of observer for general time-varying systems of the form Equations A-1 - A-4. This different state reconstructor has <u>reduced dimension</u> $(n + \rho - m)$, where m is the rank of the output matrix C. The recipe for this reduced-dimension state reconstructor may be stated as follows: The estimates $\hat{x}$ of the plant state and $\hat{z}$ of the disturbance state are given by the algebraic "assembly" equations

$$\hat{x} = \left[C^T \, (CC^T)^{-1} - T_{12} \, \Sigma \right] \, y + T_{12} \, \xi \tag{A-6}$$

$$\hat{z} = T_{22} \, (\xi - \Sigma \, y) \tag{A-7}$$

where the auxiliary variable $\xi(t)$ is generated on-line by the $(n + \rho - m)$ - degree system (dynamic filter)

$$\dot{\xi} = (\mathcal{D} + \Sigma \, \mathcal{H}) \, \xi + \Psi \, y + \Omega \, u \quad . \tag{A-8}$$

The filter (Equation A-8) is driven by the plant output y and the control u, and the matrices in Equations A-6 - A-8 are defined as follows:

a) The matrices T_{12} and T_{22} have dimensions $n \times (n + \rho - m)$ and $\rho \times (n + \rho - m)$, respectively, and are chosen to satisfy

$$\left[C \mid O \right] \left[\frac{T_{12}}{T_{22}} \right] \equiv O \; ; \; \text{rank} \left[\frac{T_{12}}{T_{22}} \right] \equiv n + \rho - m \qquad \text{(A-9)}$$

It is remarked that T_{12} and T_{22} always exist, are not unique, and are readily computed.

b) The elements of the matrix Σ are chosen to satisfy certain stability specifications, as will be seen in the example to follow.

c)

$$\mathcal{S} = \left[T_{12} \mid T_{22} \right] \left(\left[\frac{A}{O} \middle| \frac{FH}{D} \right] \left[\frac{T_{12}}{T_{22}} \right] - \left[\frac{\dot{T}_{12}}{\dot{T}_{22}} \right] \right) \qquad \text{(A-10)}$$

$$\mathcal{K} = \left[C \mid O \right] \left(\left[\frac{A}{O} \middle| \frac{FH}{D} \right] \left[\frac{T_{12}}{T_{22}} \right] - \left[\frac{\dot{T}_{12}}{\dot{T}_{22}} \right] \right) \qquad \text{(A-11)}$$

d)

$$\Psi = (\bar{T}_{12} + \Sigma C) (A C^{\# T} - \dot{C}^{\# T}) - (\mathcal{S} + \Sigma \mathcal{K}) \Sigma + \dot{\Sigma} \qquad \text{(A-12)}$$

$$\Omega = (\bar{T}_{12} + \Sigma C) B \tag{A-13}$$

$$C^{\#} = (CC^T)^{-1} C \tag{A-14}$$

e)

$$\bar{T}_{12} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{12}^T \tag{A-15}$$

$$\bar{T}_{22} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{22}^T \tag{A-16}$$

The reduced-dimension state reconstructor Equations (A-6) through (A-8) produce estimates which have estimation errors $\varepsilon_x = x - \hat{x}$ and $\varepsilon_z = z - \hat{z}$ given by

$$\varepsilon_x = x - \hat{x} = T_{12}\, \varepsilon \qquad \text{(A-17)}$$

$$\varepsilon_z = z - \hat{z} = T_{22}\, \varepsilon \qquad \text{(A-18)}$$

where the error variable ε is governed by

$$\dot{\varepsilon} = (\mathcal{D} + \Sigma\, \mathcal{H})\, \varepsilon + \bar{T}_{22}\, \sigma\,(t) \qquad \text{(A-19)}$$

where, as mentioned before, Σ is chosen to satisfy the stability requirement that $\varepsilon(t) \to o$ promptly, from any initial condition $\varepsilon\,(t_o)$.

A.2 The Design Example

In this section we consider the design of a reduced-dimension composite-state reconstructor for the special case where the system matrices have constant elements and are specified as

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{A-20}$$

$$B = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A-21}$$

$$C = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{A-22}$$

$$D = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \tag{A-23}$$

$$F = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{A-24}$$

$$H = \begin{bmatrix} 1 & 0 \end{bmatrix} \tag{A-25}$$

This problem has

$$n = \dim A = 2$$

$$p = \dim D = 2$$

$$m = \text{rank } C = 1$$

Hence the reduced-dimension observer has dimension $(n + \rho - m) = 3$.

The matrices T_{12} and T_{22} are respectively, any $n \times (n + \rho - m) = 2 \times 3$ matrix, and any $\rho(n + \rho - m) = 2 \times 3$ matrix satisfying

$$\left[C \;\middle|\; O\right] \left[\frac{T_{12}}{T_{22}}\right] \equiv O \; ; \; \text{rank} \left[\frac{T_{12}}{T_{22}}\right] = 3 \qquad \text{(A-26)}$$

We will choose

$$T_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \qquad \text{(A-27)}$$

$$T_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad \text{(A-28)}$$

It is readily verified that

$$\left[\begin{array}{cc|cc} 1 & 0 & 0 & 0 \end{array}\right] \left[\begin{array}{ccc} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{array}\right] \equiv 0 \qquad \text{(A-29)}$$

$$\text{rank}\begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ \hline 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 3 \tag{A-30}$$

and therefore A-26 is satisfied. Then we have

$$\bar{T}_{12} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{12}^T = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{A-31}$$

$$\bar{T}_{22} = (T_{12}^T T_{12} + T_{22}^T T_{22})^{-1} T_{22}^T = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{A-32}$$

Next, T_{12} and T_{22} are used in Equations (A-10) and (A-11) to find $\mathscr{D}$ and $\mathscr{K}$. The result is

$$\mathscr{D} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \tag{A-33}$$

$$\mathscr{K} = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix} \tag{A-34}$$

The characteristic matrix of the error Equation (A-19) is

$$(\mathcal{D} + \Sigma \mathcal{K}) = \begin{bmatrix} \Sigma_1 & 1 & 0 \\ \Sigma_2 & 0 & 1 \\ \Sigma_3 & 0 & 0 \end{bmatrix} \tag{A-35}$$

which has eigenvalues which satisfy

$$\lambda^3 - \Sigma_1 \lambda^2 - \Sigma_2 \lambda - \Sigma_3 = 0 \tag{A-36}$$

which may be written as (note: the Σ are all real)

$$(\lambda + a)\ (\lambda^2 - b\lambda - c) = 0 \tag{A-37}$$

The roots of Equation (A-36) may be written

$$\lambda_1 = -a \quad ; \ a \geq 0 \tag{A-38}$$

$$\lambda_2\ ,\ \lambda_3 = \frac{b}{2} \mp \frac{1}{2}\sqrt{b^2 + 4c} \quad ; \ b \leq 0 \tag{A-39}$$

where

$$b - a = \Sigma_1 \tag{A-40}$$

$$a\ b + c = \Sigma_2 \tag{A-41}$$

$$a\ c = \Sigma_3 \tag{A-42}$$

For this problem, the design parameters a and b are chosen to give eigenvalues with relatively large negative real parts in order to obtain fast settling times for the estimates. The following values were chosen:

$$a = 30 \tag{A-43}$$

$$b = -42 \tag{A-44}$$

$$c = -541 \tag{A-45}$$

The corresponding eigenvalues are

$$\lambda_1 = -30 \tag{A-46}$$

$$\lambda_2\ ,\ \lambda_3 = -21 \mp j\ 10 \tag{A-47}$$

which yield the values

$$\Sigma_1 = b - a = -72 \tag{A-48}$$

$$\Sigma_2 = ab+c = -1801 \tag{A-49}$$

$$\Sigma_3 = ac = -16230 \tag{A-50}$$

Next, Equation (A-12) is used to find

$$\Psi = -\begin{pmatrix} \Sigma_1^2 + \Sigma_2 \\ \Sigma_1 \Sigma_2 + \Sigma_3 \\ \Sigma_1 \Sigma_3 \end{pmatrix} = \begin{pmatrix} -\quad 3,383 \\ -\quad 113,442 \\ -\ 1,168,560 \end{pmatrix} \tag{A-51}$$

and Equation (A-13) is used to compute

$$\Omega = \begin{bmatrix} \Sigma_1 & 1 \\ \Sigma_2 & 0 \\ \Sigma_3 & 0 \end{bmatrix} \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \tag{A-52}$$

The differential equation governing the auxiliary variable ξ is thus obtained as

$$\begin{pmatrix} \dot{\xi}_1 \\ \dot{\xi}_2 \\ \dot{\xi}_3 \end{pmatrix} = \begin{bmatrix} \Sigma_1 & 1 & 0 \\ \Sigma_2 & 0 & 1 \\ \Sigma_3 & 0 & 0 \end{bmatrix} \begin{pmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \end{pmatrix} + \begin{pmatrix} -\Sigma_1^2 - \Sigma_2 \\ -\Sigma_1 \Sigma_2 - \Sigma_3 \\ -\Sigma_1 \Sigma_3 \end{pmatrix} y \tag{A-53}$$

$$+ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} u$$

where Σ_1, Σ_2 and Σ_3 are specified in Equations (A-48) through (A-50).

When the parameters of this example are substituted into Equations (A-6) and (A-7) the state estimates become

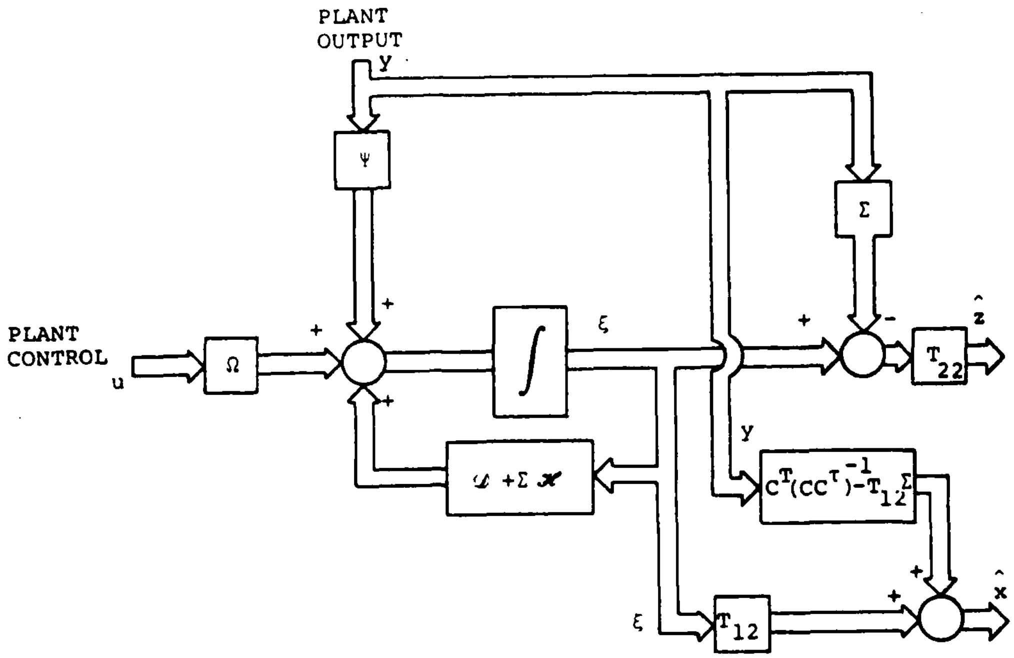
$$\hat{x} = \begin{pmatrix} 1 \\ -\Sigma_1 \end{pmatrix} y + \begin{pmatrix} 0 \\ \xi_1 \end{pmatrix} \tag{A-54}$$

$$\hat{z} = \begin{pmatrix} \xi_2 \\ \xi_3 \end{pmatrix} - \begin{pmatrix} \Sigma_2 \\ \Sigma_3 \end{pmatrix} y \tag{A-55}$$

where the values of Σ_1, Σ_2 and Σ_3 are specified in Equations (A-48) through (A-50) and ξ_1, ξ_2 and ξ_3 are the solutions of Equation (A-53).

A block diagram of the final state reconstructor is shown in Figure A-1.

If the gain values in (A-48) - (A-51) are judged to be too large, they can be reduced by making appropriate reductions in the (magnitudes of the) real parts of the choosen roots (A-46), (A-47).



<u>Parameter values:</u>

$$\Psi = \begin{pmatrix} -3,383 \\ -113,442 \\ -1,168,560 \end{pmatrix} ; \quad \Sigma = \begin{pmatrix} -72 \\ -1,801 \\ -16,230 \end{pmatrix}$$

$$\Omega = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} ; \quad \left(C^T(CC^T)^{-1} - T_{12}\,\Sigma\right) = \begin{pmatrix} 1 \\ 72 \end{pmatrix}$$

$$T_{12} = \begin{bmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} ; \quad T_{22} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$(\mathscr{D} + \Sigma\,\mathscr{H}) = \begin{bmatrix} -72 & 1 & 0 \\ -1,801 & 0 & 1 \\ -16,230 & 0 & 0 \end{bmatrix}$$

Figure A-1. Reduced-order state reconstructor for the example design.

APPENDIX B

DIGITAL PROGRAMS FOR ANALYSIS OF DISTURBANCE-UTILIZING CONTROL SYSTEMS

B.1 Introduction.

This appendix describes the digital computer programs used in Chapters III and IV to solve for the disturbance-utilizing control and to obtain data used in the analysis of its performance. The basic program, called DUCAT (Disturbance-Utilizing Control Analysis Technique), has three versions: DUCAT1--for scalar plant/scalar disturbance/scalar set-point problems, DUCAT2--for zero set-point problems with a second-order plant and a second-order disturbance model, and DUCAT3--for zero set-point problems with a fourth-order plant and fourth-order disturbance model (or two second-order disturbance models).

The DUCAT program can implement either a disturbance-utilizing control law or, for comparison, a conventional linear-quadratic control law for a time-invariant: plant (2.1), (2.2), disturbance-command models (2.3), (2.4), (2.6), (2.7) and performance index (2.5). The time varying gains are obtained by solving the matric differential equations using Runge-Kutta fourth-order integration via ACSL

(Advanced Continuous Simulation Language) on a CDC-6600 computer. ACSL is also used for integration of the plant differential equations. Initial conditions for the forward integrations of the gain equations are found by performing backward-time integrations of these equations, starting at a specified terminal time T, with the known terminal conditions, and integrating back to the time $t_o = o$. The values of the gains at $t = o$ are then stored, to be used as the initial conditions for the subsequent forward-time runs. This procedure is used to avoid the storage requirements associated with storing the gain time-functions for all $o \leq t \leq T$.

An important feature of DUCAT is the capability of solving for the "T-minimin" values of the performance index J, in either the disturbance-utilizing or conventional linear-quadratic problem. The values of $J[T_i]$ are obtained for a selected set of values of specified terminal times T_i in some specified interval $[T_{min}, T_{max}]$. The program then selects the minimum value J_{min} among that set and displays the optimal control, state trajectory and related parameters for the particular T_{min} corresponding to J_{min}.

The flow of the basic DUCAT program, which describes all three program versions, is presented in Figures B-1 and B-2. Listings of the three program versions are contained in the following sections.

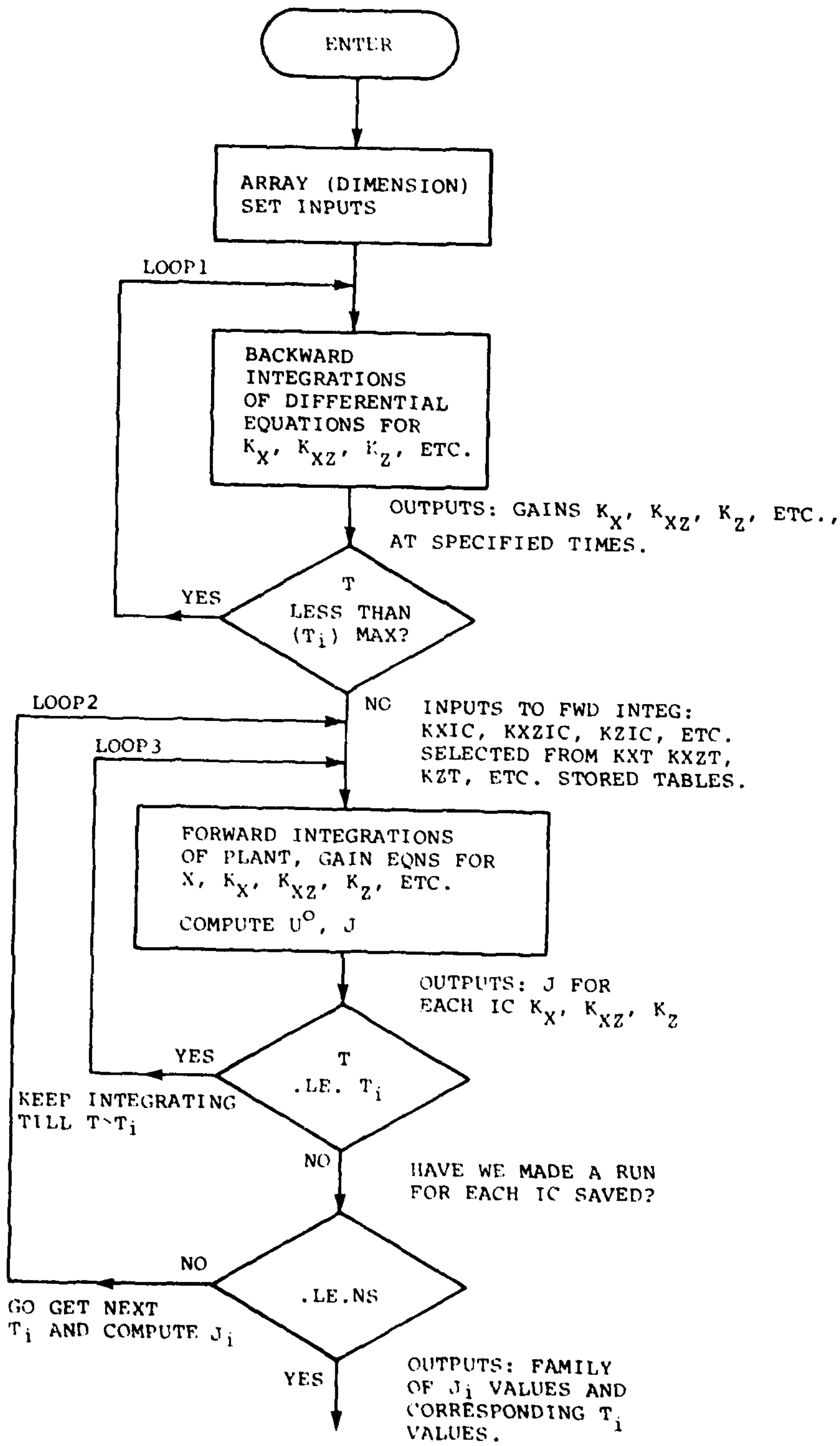


Figure B-1. Flow of DUCAT program, first segment.

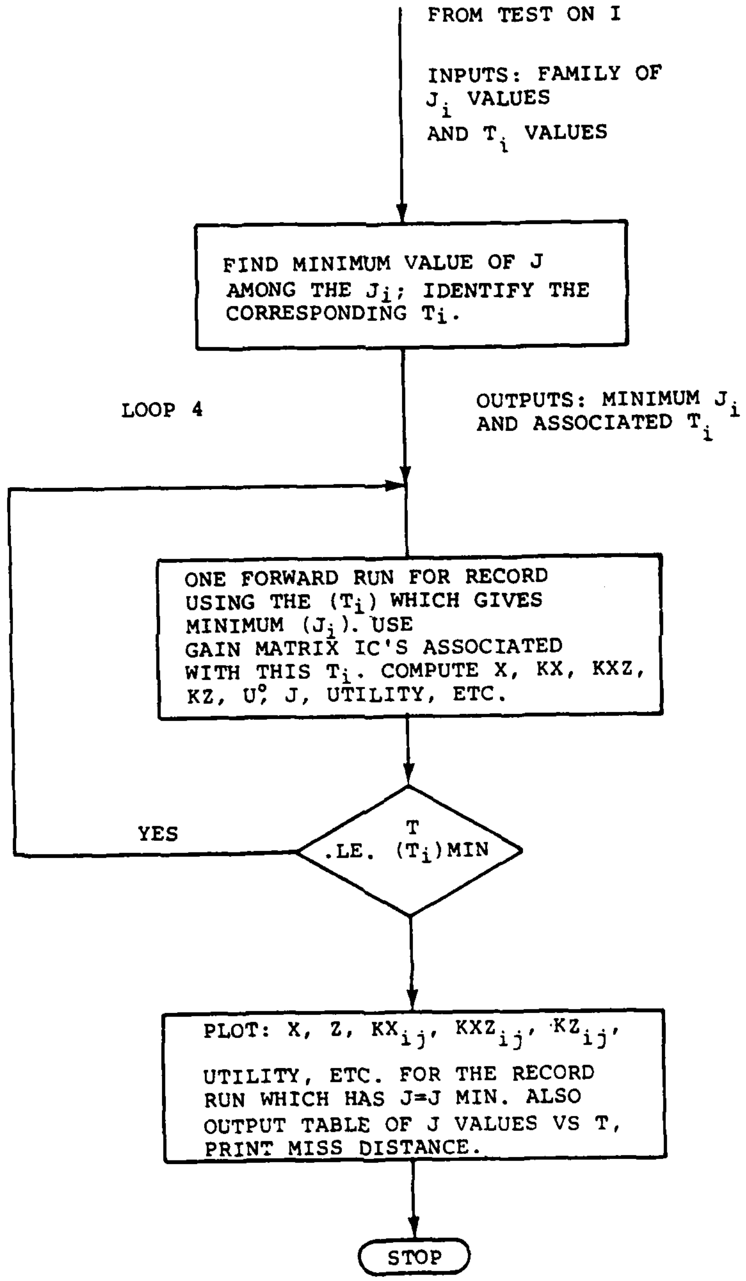


Figure B-2. Flow of DUCAT program, final segment.

B.2 Program DUCAT1

This section contains the listing and parameter definitions for the DUCAT1 program, which solves for a disturbance-utilizing control or a conventional linear-quadratic control in regulator problems involving scalar plant/scalar disturbance/scalar set-point. Non-zero set-point or zero set-point regulator problems may be solved by DUCAT1. Input parameters (determined by "SET" statements near the end of the program), and plot output parameters (identified in the "PREPAR" statement at the end of the program) are defined below. In addition, the printed output parameters are also defined.

INPUT PARAMETERS

PARAMETER	DEFINITION
ALPHAS	α as defined in Chapter III.
AS	A as defined in Chapter III.
BS	B as defined in Chapter III.
CINTFD	Forward-time data communication interval (sec).
CINTBD	Backward-time data communication interval (sec).
CS	C as defined in Chapter III.
CSETS	Set-point input (ft).

C1S	C_1 as defined in Chapter III (ft/sec^2).
DTBCK	Integration interval, backward-time (sec).
DTFWD	Integration interval, forward-time (sec).
ES	E as defined in Chapter III.
FS	F as defined in Chapter III.
GS	G as defined in Chapter III.
HS	H as defined in Chapter III.
IALG	Logic input. Selects fourth-order Runge-Kutta integration when IALG = 5.
QS	Q as defined in Chapter III.
RS	R as defined in Chapter III.
SS	S as defined in Chapter III.
TFINIT	Maximum value T_{max} for final-time scan (sec).
XSIC	Initial condition of the state x (ft).

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PLOT OUTPUT PARAMETERS

PARAMETER	DEFINITION
T	Program time (sec).
XSP	x as defined in Chapter III.
ZSP	z as defined in Chapter III.
KXSP	k_x as defined in Chapter III.
KXCSP	k_{xc} as defined in Chapter III.
KXZSP	k_{xz} as defined in Chapter III.
KCSP	k_c as defined in Chapter III.
KCZSP	k_{cz} as defined in Chapter III.
KZSP	k_z as defined in Chapter III.
WSP	w as defined in Chapter III.
VSP	Unused.
UOPTSP	u° as defined in Chapter III.
BURDSP	Burden as defined in Chapter III.
ASSISP	Assistance as defined in Chapter III.
UTILSP	Utility as defined in Chapter III.
THXZSP	θ_{xz} as defined in Chapter III.
THCZSP	θ_{cz} as defined in Chapter III.

PRINTED OUTPUT PARAMETERS

PARAMETER	DEFINITION
KXST	Stored initial-condition of k_x.
KXCST	Stored initial-condition of k_{xc}.
KXZST	Stored initial-condition of k_{xz}.
KCST	Stored initial-condition of k_c.
KCZST	Stored initial-condition of k_{cz}.
KZST	Stored initial-condition of k_z.
TAUTAS	A particular specified terminal time T_i.
JMATAB	Value of $\frac{1}{2} e^T(T)\ s\ e(T)$.
J2TAB	Value of $\frac{1}{2} \int_0^T e^T(t)\ qe(t)\ dt$
J3TAB	Value of $\frac{1}{2} \int_0^T r\ u^2(t)\ dt$
JTFTAS	J (T)
XTABS	x(T).

The listing of Program DUCAT1 follows:

DUCATI

```
PROGRAM OPTG
COMMENT     COMPUTE OPTIMAL CONTROL FOR CASE OF MISSILE INTERCEPT      ...
C           WITH DISTURBANCES PRESENT                                  ...
C           ALSO COMPUTE THE MINIMUM J IN A FAMILY OF J S              ...
C           THEN COMPUTE ASSISTANCE BURDEN AND UTILIZATION AND PLOT    ...
C
INITIAL
      ARRAY KXST(100),KXCST(100),KXZST(100)
      ARRAY KCST(100),KCZST(100),KZST(100)
      ARRAY XTABS(100)
      ARRAY JMATAB(100),J2TAB(100),J3TAB(100)
      ARRAY JTFTAS(100)
      ARRAY TAUTAS(100)

      INTEGER I,J
      INTEGER NS

      CONSTANT DTFWD=0.,TFINIT=0.,DTBCK=0.
      CONSTANT AS=0.,BS=0.,CS=0.,CSETS=0.,DS=0.,FS=0.
      CONSTANT C1S=0.,ALPHAS=0.
      CONSTANT ES=0.
      CONSTANT GS=0.,HS=0.,QS=0.,RS=0.,SS=0.
      CONSTANT KXS=0.,KXCS=0.,KXZS=0.,KCS=0.,KCZS=0.,KZS=0.
      CONSTANT XSIC=0.
      CONSTANT WSP=0., BURDSP=0., ASSISP=0., UTILSP=0.

      LOGICAL FWDBWD,LAST

      NS=0
      FHS=FS*HS
      BSOR=(BS**2)/RS
      KXSIC=(CS**2)*SS
      KXCSIC=-CS*SS*GS
      KXZSIC=0.
      KCSIC=(GS**2)*SS
      KCZSIC=0.
      KZSIC=0.
      FWDBWD=.FALSE.
      LAST=.FALSE.
LOOP..CONTINUE
      CONSTANT CINTLF=0.02
      CONSTANT CINTFD=0.1,CINTBD=0.1
      MAXT=RSW(FWDBWD,DTFWD,DTBCK)
      TSTP = 100.
      JOPTS=0.
      BURDES=0.
      ASSISS=0.
      THXZS=0.
      THCZS=0.
      UTILS=0.
      WS=0.
```

```
      ZS=0.
      KXSP=0.
      KXCSP=0.
      KXZSP=0.
      KCSP=0.
      KCZSP=0.
      KZSP=0.
      THXZSP=0.
      THCZSP=0.
      XSP=0.
      ZSP=0.
      UOPTSP=0.
      JMAYES=0.
      J2=0.
      J3=0.
END  $  "INITIAL"
DYNAMIC
      CINTERVAL  CINT=0.1
      MAXTERVAL  MAXT=0.001
      NSTEPS  NSTP=1
      CINTFL=RSW(LAST,CINTLF,CINTFU)
      CINT=RSW(FWDBWD,CINTFL,CINTBD)
      VARIABLE  T=0.
ALGORITHM  IALG=4
      IF (FWDBWD) GO TO SKPSV
      IF(T.LE.0.) GO TO SKPSV
      NS=NS+1
      KXST(NS)=KXS
      KXCST(NS)=KXCS
      KXZST(NS)=KXZS
      KCST(NS)=KCS
      KCZST(NS)=KCZS
      KZST(NS)=KZS
SKPSV..CONTINUE
      TERMT(ABS(T-TFINIT).LE.CINTBD.AND..NOT.FWDBWD)
      IF(FWDBWD) TSTP=CINTBD*FLOAT(NS)
      IF(FWDBWD) CINT=AMIN1(CINTFL,TSTP-T)
      CONSTANT TGOMN=1.E-3
TERMT(ABS(T-TSTP).LE.TGOMN.AND.FWDBWD)
DERIVATIVE  OPTGN
      CALL FTNDRV
      KXS=INTEG(KXSD,KXSIC)
      KXCS=INTEG(KXCSD,KXCSIC)
      KXZS=INTEG(KXZSD,KXZSIC)
      KCS=INTEG(KCSD,KCSIC)
      KCZS=INTEG(KCZSD,KCZSIC)
      KZS=INTEG(KZSD,KZSIC)
      XS=INTEG(XSD,XSIC)
      J2=INTEG(J2DOT,0.)
      J3=INTEG(J3DOT,0.)
      THXZS=ATAN2(-2.*KXZS,(KZS+1.E-30))*57.29578
      THCZS=ATAN2(-2.*KCZS,(KZS+1.E-30))*57.29578
END  $  "DERIVATIVE"
END  $  "DYNAMIC"
TERMINAL
```

```
      IF (FWDBWD.AND.LAST) GO TO TERM
      IF (FWDBWD) GO TO CYCLE
      WRITE(6,980)(KXST(I),KXCST(I),KXZST(I),KCST(I),KCZST(I),KZST(I)...
      ,I=1,NS)
980.. FORMAT(T10,6E15.5)
      INTEGER NSFNL
      NSFNL=NS
      NS=0
      FWDBWD=.TRUE.
      GO TO SETIC
CYCLE..CONTINUE
      JMAYES=0.5*(SS*CS*CS*XS*XS-2.*SS*CS*GS*CSETS*XS...
      +SS*GS*GS*CSETS*CSETS)
      JTFS=JMAYES+J2+J3
      JMATAB(NS)=JMAYES
      J2TAB(NS)=J2
      J3TAB(NS)=J3
      XTABS(NS)=XS
      JTFTAS(NS)=JTFS
      TAUTAS(NS)=T

SETIC..NS=NS+1
      IF (NS.GT.NSFNL) GO TO FNLRUN
RECOR..CONTINUE
      KXSIC=KXST(NS)
      KXCSIC=KXCST(NS)
      KXZSIC=KXZST(NS)
      KCSIC=KCST(NS)
      KCZSIC=KCZST(NS)
      KZSIC=KZST(NS)
LOG

      GO TO LOOP
FNLRUN..CONTINUE
      JZEROS=1.E30
      DO JMINS I=1,NSFNL
      IF(JZEROS.LT.JTFTAS(I)) GO TO JMINS
      JZEROS=JTFTAS(I)
      TFZERS=TAUTAS(I)
      NS=I
JMINS.. CONTINUE
      NS=6
      WRITE(6,199) (TAUTAS(I),JMATAB(I),J2TAB(I),J3TAB(I),    ...
         JTFTAS(I),XTABS(I),I=1,NSFNL)
199.. FORMAT (T10,F8.2,T20,E10.4,T35,E10.4,T50,E10.4,    ...
         T65,E10.4,T80,E10.4)
      REWIND 8
      LAST=.TRUE.
      GO TO RECOR
TERM..CONTINUE
END   $  "TERMINAL"
END   $  "PROGRAM"
```

```
      SUBROUTINE FTNDRV
$
       DS=-ALPHAS
      IF (FWDBWD) GO TO 1003
C        GET DERIVATIVES FOR BACKWARD INTEGRATIONS
      KXSD=(AS-BSOR*KXS)*KXS+KXS*AS+(CS*CS)*QS
      KXCSD=(AS-BSOR*KXS)*KXCS+KXCS*ES-CS*QS*GS
      KXZSD=(AS-BSOR*KXS)*KXZS+KXS*FHS+KXZS*DS
      KCSD=2.*KCS*ES-BSOR*(KXCS*KXCS)+(GS*GS)*QS
      KCZSD=KCZS*DS+ES*KCZS-KXCS*(BSOR*KXZS-FHS)
      KZSD=2.*KZS*DS-BSOR*(KXZS*KXZS)+2.*FHS*KXZS
      J2DOT=0.
      J3DOT=0.
      XSD=0.
      RETURN
C
C         LOOP TO COMPUTE X,KX,KXZ,KZ,UOPT,J
 1003 CONTINUE
C
      XSD1=AS*XS
      XSD2=BS*UOPTS
C        GET DISTURBANCES,MISSILE AND TARGET
      WS=C1S*EXP(-ALPHAS*T)
      XSD3=FS*WS
      XSD=XSD1+XSD2+XSD3
      ZS=WS
      KXSD=(-AS+BSOR*KXS)*KXS-KXS*AS-(CS*CS)*QS
      KXCSD=(-AS+BSOR*KXS)*KXCS-KXCS*ES+CS*QS*GS
      KXZSD=(-AS+BSOR*KXS)*KXZS-KXS*FHS-KXZS*DS
      KCSD=-2.*KCS*ES+BSOR*(KXCS*KXCS)-(GS*GS)*QS
      KCZSD=-(KCZS*DS+ES*KCZS)+KXCS*(BSOR*KXZS-FHS)
      KZSD=-2.*KZS*DS+BSOR*(KXZS*KXZS)-2.*FHS*KXZS
C        COMPUTE UOPT
      DACON=1.
      UOPTS=-(BS/RS)*(KXS*XS+KXCS*CSETS+KXZS*ZS*DACON)
C        COMPUTE PERFORMANCE INDEX AT T.LE.TF
      J2DOT=0.5*(QS*CS*CS*XS*XS-2.*QS*CS*GS*XS*CSETS
     C+QS*GS*GS*CSETS*CSETS)
      J3DOT=0.5*(UOPTS*UOPTS*RS)
      IF (.NOT.LAST) RETURN
C        COMPUTE BURDEN, ASSISTANCE AND UTILIZATION IN RECORD RUN
      BURDES=0.5*KZS*ZS*ZS
      ASSISS=-KXZS*XS*ZS-KCZS*CSETS*ZS
      UTILS=ASSISS-BURDES
      BURDSP=BURDES
      ASSISP=ASSISS
      UTILSP=UTILS
      WSP=WS
      KXSP=KXS
      KXCSP=KXCS
      KXZSP=KXZS
      KCSP=KCS
      KCZSP=KCZS
      KZSP=KZS
```

```
      THXZSP=THXZS
      THCZSP=THCZS
      UOPTSP=UOPTS
      VSP=VS
      XSP=XS
      ZSP=ZS
      END
```

TRANSLATION TIME =   4.303*

```
SET ALPHAS=0.1
SET AS=1.
SET BS=1.
SET CINTFD=1.
SET CINTBD=1.
SET CS=1.
SET CSETS=-10.
SET C1S=-16.1
SET DTBCK=0.02
SET DTFWD=0.02
SET ES=0.
SET FS=1.
SET GS=1.
SET HS=1.
SET IALG=5
SET QS=0.
SET RS=1.
SET SS=1.
SET TFINIT=6.
SET XSIC=30.
PREPAR T,XSP,ZSP,KXSP,KXCSP,KXZSP,KCSP,KCZSP,KZSP,....
WSP,VSP,UOPTSP,BURDSP,ASSISP,UTILSP,THXZSP,THCZSP
```

B.3 Program DUCAT2.

This section contains a listing and parameter definitions for the DUCAT2 program which solves for a disturbance-utilizing control or a conventional linear-quadratic control in the case of zero set-point (homing intercept) regulator problems with a second-order plant and a second-order disturbance model. Input parameters determined by "SET" statements, and plot output parameters identified by "PREPAR" statements at the end of the program, are defined below. Printed output parameters are also defined.

INPUT PARAMETERS

PARAMETER	DEFINITION
A	A as defined in Chapter IV.
ALH	α_h as defined in Chapter IV.
AT	Transpose of A.
B	B as defined in Chapter IV.
BBT	BB^T.
C	C as defined in Chapter IV.
CINTBD	Backward-time communication interval (sec).

CINTFD	Forward-time communication interval (sec).
D	D as defined in Chapter IV.
DTBCK	Backward-time integration interval (sec).
DTFWD	Forward-time integration interval (sec).
F	F as defined in Chapter IV.
FH	FH as defined in Chapter IV.
H	H as defined in Chapter IV.
IALG	Logic input. Selects Fourth-Order Runge-Kutta integration when IALG = 5.
KX	K_x as defined in Chapter IV.
KXZ	K_{xz} as defined in Chapter IV.
KZ	K_z as defined in Chapter IV.
Q	Q as defined in Chapter IV.
RINV	Inverse of R, Chapter IV.
S	S as defined in Chapter IV.
TFINIT	Maximum value T_{max} for final-time scan (sec).
VMR	Missile velocity along LOS (ft/sec).
VTNIC	Initial value, velocity of missile normal to LOS (ft/sec).
VTR	Target velocity along LOS (ft/sec).
WMXTB	Missile disturbance input table, independent variable.
WMYTB	Missile disturbance input table, dependent variable.

WTXTB	Target disturbance input table, independent variable.
WTYTB	Target disturbance input table, dependent variable.
WDMXTB	Unused.
WDTYTB	Unused.
WDTXTB	Unused.
XIC	Initial-condition value of x vector.
XMRIC	Initial-condition value of missile position along LOS (ft).
XTRIC	Initial-condition value of target position along LOS (ft).
XTNIC	Initial-condition value of target position normal to LOS (ft).

PLOT OUTPUT PARAMETERS

PARAMETER	DEFINITION
T	Program time (sec).
XP	x as defined in Chapter IV.
ZP	z as defined in Chapter IV.
KXP	K_x as defined in Chapter IV.
KXZP	K_{xz} as defined in Chapter IV.
KZP	K_z as defined in Chapter IV.
W	w as defined in Chapter IV.
WT	Net target disturbance (ft/sec^2).

WM	Net missile disturbance (ft/sec).
BURDEN	Burden as defined in Chapter IV.
ASSIST	Assistance as defined in Chapter IV.
UTIL	Utility as defined in Chapter IV.
JZERO	Minimum value of J in a set of J_i values.
TFZERO	Value of T corresponding to JZERO.
UOPTP	u° as defined in Chapter IV.
XTNP	Target position normal to LOS (ft).
XTRP	Target position along LOS (ft).
XMNP	Missile position normal to LOS (ft).
XMRP	Missile position along LOS (ft).
THMVP	Missile velocity vector angle, degrees.
XTP	Target position, horizontal (ft).
YTP	Target position, vertical (ft).
XMP	Missile position, horizontal (ft).
YMP	Missile position, vertical (ft).
XMD	Missile velocity, horizontal (ft/sec).
YMD	Missile velocity, vertical (ft/sec).
XML	Unused.
YML	Unused.

PRINTED OUTPUT PARAMETERS

PARAMETER	DEFINITION
KXT	Stored initial-condition of K_x.
KXZT	Stored initial-condition of K_{xz}.
KZT	Stored initial-condition of K_z.
TAUTAB	A particular specified terminal time T_i.
JMATAB	Value of $\frac{1}{2}\, x^T(T)\, Sx(T)$
J2TAB	Value of $\frac{1}{2} \int_0^T x^T(t)\, Qx(t)\, dt$
J3TAB	Value of $\frac{1}{2} \int_0^T u^T(t)\, Ru(t)\, dt$
JTFTAB	Value of J at $t = T$.
X1TAB	$x_1(T)$.
X2TAB	$x_2(T)$.
EAUTAB	EAU as defined in Chapter IV.
EUTAB	EU as defined in Chapter IV.

The listing of Program DUCAT2 follows:

DUCAT2

```
PROGRAM OPTG
COMMENT    COMPUTE OPTIMAL CONTROL FOR CASE OF MISSILE INTERCEPT     ...
C          WITH DISTURBANCES PRESENT                                 ...
C          ALSO COMPUTE THE MINIMUM J IN A FAMILY OF J S             ...
C          THEN COMPUTE ASSISTANCE BURDEN AND UTILIZATION AND PLOT   ...
C
INITIAL
      ARRAY    A(2,2),AT(2,2),B(2,1),BT(1,2),BBT(2,2),C(1,2),D(2,2)
      ARRAY    F(2,1),H(1,2),FH(2,2),KX(2,2),KXIC(2,2),KXZ(2,2)
      ARRAY    KXDOT(2,2),KXZDOT(2,2),KZDOT(2,2),X(2,1),Z(2,1),Q(2,2)
      ARRAY    S(2,2),V1(2,1),V2(2,1),V3(2,1),XDOT1(2,1),XDOT2(2,1)
      ARRAY    XDOT3(2,1),XDOT(2,1),XIC(2,1),KXT(2,2,100)
      ARRAY    KXZT(2,2,100),KZT(2,2,100),WMXTB(6),WMYTB(6),WTXTB(6)
      ARRAY    WTYTB(6),XT(1,2),ZT(1,2),V4(2,1),ARA(2,2),ARB(2,2)
      ARRAY    KXBBT(2,2),ARC(2,2),ARD(2,2),ARE(2,2),ARF(2,2)
      ARRAY   ARG(2,2),ARH(2,2),ARHT(2,2),BBTKXZ(2,2),ARI(2,2),ARJ(2,2)
      ARRAY    ARJT(2,2),TAUTAB(100),SX(2,1),JTFTAB(100)
      ARRAY    KZ(2,2),KXZIC(2,2),KZIC(2,2),KXZTR(2,2)
      ARRAY    WDMYTB(10),WDMXTB(10),WDTYTB(6),WDTXTB(6)
      ARRAY    KXP(4),KXZP(4),KZP(4)
      ARRAY    XP(2),ZP(2),X1TAB(100),X2TAB(100)
      ARRAY V5(2,1)
      ARRAY JMATAB(100),J2TAB(100),J3TAB(200),EAUTAB(100),EUTAB(100)


      INTEGER I,J
      INTEGER NS


      CONSTANT A=4*0.,B=2*0.,C=2*0.,D=4*0.,F=2*0.
      CONSTANT H=2*0.,Q=4*0.,RINV=0.,S=4*0.,DTBCK=0.
      CONSTANT DTFWD=0.,TFINIT=0.,KX=4*0.
      CONSTANT KXZ=4*0.,KZ=4*0.,WMXTB=6*0.,WMYTB=6*0.
      CONSTANT W=0.,WD=0.
      CONSTANT WTXTB=6*0.,WTYTB=6*0.
      CONSTANT XIC=2*0.
      CONSTANT XMRIC=0.,XTRIC=0.,VMR=0.,VTR=0.,VMN=0.,VTN=0.
      CONSTANT VMNIC=0.,VTNIC=0.,XMNIC=0.,XTNIC=0.
      CONSTANT WDMYTB=10*0.,WDMXTB=10*0.,WDTYTB=6*0.,WDTXTB=6*0.
      CONSTANT ALH=0.


      LOGICAL FWDBWD,LAST


      NS=0
      DO BWDIC I=1,2
      DO BWDIC J=1,2
      KXIC(I,J)=S(I,J)
      KXZIC(I,J)=0.
      KZIC(I,J)=0.
BWDIC..CONTINUE
      CALH=COS(ALH*0.01745)
      SALH=SIN(ALH*0.01745)
      FWDBWD=.FALSE.
```

```
      LAST=.FALSE.
LOOP..CONTINUE
      CONSTANT CINTLF=0.02
      CONSTANT CINTFD=0.1,CINTBD=0.1
      MAXT=RSW(FWDBWD,DTFWD,DTBCK)
      UOPT=0.0
      BURDEN = 0.0
      ASSIST = 0.0
      UTIL = 0.0
      W = 0.0
      WD=0.
      WT = 0.0
      WM = 0.0
      CONSTANT Z=2*0.
      TSTP = 100.
      KXP(1)=0.0
      KXP(2)=0.0
      KXP(3)=0.0
      KXP(4)=0.0
      KXZP(1)=0.0
      KXZP(2)=0.0
      KXZP(3)=0.0
      KXZP(4)=0.0
      KZP(1)=0.0
      KZP(2)=0.0
      KZP(3)=0.0
      KZP(4)=0.0
      XP(1)=0.0
      XP(2)=0.0
      ZP(1)=0.0
      ZP(2)=0.0
      UOPTP=0.0
      J2=0.
      J3=0.
      EUTAB=0.
      EAUTAB=0.
      J2S=0.
      J2P=0.
      J3P=0.
      EUP=0.
      EAUP=0.
      JMAYER=0.
      WL=0.
      TL=0.
      XTNP=0.
      XTRP=0.
      XMNP=0.
      XMRP=0.
      XTP=0.
      YTP=0.
      XMP=0.
      YMP=0.
      XML=0.
      YML=0.
      THMVP=0.
```

```
      XMD=0.
      YMD=0.
END  $  "INITIAL"
DYNAMIC
      CINTERVAL  CINT=0.1
      MAXTERVAL  MAXT=0.001
      NSTEPS  NSTP=1
      CINTFL=RSW(LAST,CINTLF,CINTFD)
      CINT=RSW(FWDBWD,CINTFL,CINTBD)
      VARIABLE  T=0.
ALGORITHM  IALG=4
      IF (FWDBWD) GO TO SKPSV
      IF(T.LE.0.) GO TO SKPSV
      NS=NS+1
      DO SVLOOP I=1,2
      DO SVLOOP J=1,2
      KXT(I,J,NS)=KX(I,J)
      KXZT(I,J,NS)=KXZ(I,J)
      KZT(I,J,NS)=KZ(I,J)
SVLOOP..CONTINUE
      GO TO SKPDER
SKPSV..CONTINUE
      WD=RSW(T.EQ.0.,0.,(W-WL)/(T-TL+1.E-20))
      WL=W
      TL=T
SKPDER..CONTINUE
      TERMT(ABS(T-TFINIT).LE.CINTBD.AND..NOT.FWDBWD)
      IF(FWDBWD) TSTP=CINTBD*FLOAT(NS)
      IF(FWDBWD) CINT=AMIN1(CINTFL,TSTP-T)
      CONSTANT TGOMN=1.E-3
TERMT(ABS(T-TSTP).LE.TGOMN.AND.FWDBWD)
DERIVATIVE  OPTGN
      CALL FTNDRV
      KX=INTVC(KXDOT,KXIC)
      KXZ=INTVC(KXZDOT,KXZIC)
      KZ=INTVC(KZDOT,KZIC)
      X=INTVC(XDOT,XIC)
      J2=INTEG(J2DOT,0.)
      J3=INTEG(J3DOT,0.)
      EU=INTEG(EUDOT,0.)
      EAU=INTEG(EAUDOT,0.)
      VTN=INTEG(WT,VTNIC)
      XTN=INTEG(VTN,XTNIC)
END  $  "DERIVATIVE"
END  $  "DYNAMIC"
TERMINAL
      IF (FWDBWD.AND.LAST) GO TO TERM
      IF (FWDBWD) GO TO CYCLE
      WRITE (6,98) (((KXT(II,JJ,KK),KXZT(II,JJ,KK),   ...
           KZT(II,JJ,KK),II=1,2),JJ=1,2),KK=1,NS)
98..  FORMAT(T20,3E15.5)
      INTEGER NSFNL
      NSFNL=NS
      NS=0
      FWDBWD=.TRUE.
```

```
      GO TO SETIC
CYCLE..CONTINUE
      DO JT1 I=1,2
JT1.. XT(1,I)=X(I,1)
      CALL MMPY(S,X,SX,2,2,1)
      SCM=0.0
      DO JT2 I=1,2
JT2.. SCM=SCM+XT(1,I)*SX(I,1)
      JMAYER=0.5*SCM
      JTF=JMAYER+J2+J3
      JMATAB(NS)=JMAYER
      J2TAB(NS)=J2
      J3TAB(NS)=J3
      EUTAB(NS)=EU
      EAUTAB(NS)=EAU
      X1TAB(NS)=X(1,1)
      X2TAB(NS)=X(2,1)
      JTFTAB(NS)=JTF
      TAUTAB(NS) = T

SETIC..NS=NS+1
      IF (NS.GT.NSFNL) GO TO FNLRUN
RECOR.. DO IC1 J=1,2
      DO IC1 I=1,2
      KXIC(I,J)=KXT(I,J,NS)
      KXZIC(I,J)=KXZT(I,J,NS)
      KZIC(I,J)=KZT(I,J,NS)
IC1..CONTINUE
LOG

      GO TO LOOP
FNLRUN..CONTINUE
      JZERO=1.E30
      DO JMIN I=1,NSFNL
      IF (JZERO.LT.JTFTAB(I)) GO TO JMIN
      JZERO=JTFTAB(I)
      TFZERO = TAUTAB(I)
      NS=I
JMIN..CONTINUE
      NS=9
      WRITE(6,299) (TAUTAB(I),JMATAB(I),J2TAB(I),J3TAB(I),  ...
         JTFTAB(I),X1TAB(I),X2TAB(I),EAUTAB(I),EUTAB(I),I=1,NSFNL)
299.. FORMAT(T4,F4.2,T12,E10.4,T26,E10.4,T40,E10.4,...
         T54,E10.4,T68,E10.4,T82,E10.4,T96,E10.4,T110,E10.4)
      REWIND 8
      LAST=.TRUE.
      GO TO RECOR
TERM..CONTINUE
END  $  "TERMINAL"
END  $  "PROGRAM"
```

```
      SUBROUTINE FTNORV
$
      IF (FWDBWD) GO TO 1003
      CALL MMPY (KX,A,ARA,2,2,2)
      CALL MMPY (AT,KX,ARB,2,2,2)
      CALL MMPY (KX,BBT,KXBBT,2,2,2)
      CALL MMPY (KXBBT,KX,ARC,2,2,2)
      CALL MMPY (KXBBT,KXZ,ARD,2,2,2)
      CALL MMPY (AT,KXZ,ARE,2,2,2)
      CALL MMPY (KXZ,D,ARF,2,2,2)
      CALL MMPY (KX,FH,ARG,2,2,2)
      CALL MMPY (KZ,D,ARH,2,2,2)
      CALL MMPY (BBT,KXZ,BBTKXZ,2,2,2)
      DO 5 J=1,2
      DO 5 I=1,2
    5 KXZTR(I,J)=KXZ(J,I)
      CALL MMPY(KXZTR,BBTKXZ,ARI,2,2,2)
      CALL MMPY (KXZTR,FH,ARJ,2,2,2)
      DO 6 J=1,2
      DO 6 I=1,2
      ARHT(I,J)=ARH(J,I)
    6 ARJT(I,J)=ARJ(J,I)
C
C
C        GET DERIVATIVES FOR BACKWARD INTEGRATIONS
      DO 10 J=1,2
      DO 10 I=1,2
      KXDOT(I,J)=ARA(I,J)+ARB(I,J)-ARC(I,J)*RINV+Q(I,J)
      KXZDOT(I,J)=ARG(I,J)+ARF(I,J)+ARE(I,J)-ARD(I,J)*RINV
      KZDOT(I,J) =ARJ(I,J)+ARH(I,J)+ARJT(I,J)+ARHT(I,J)-ARI(I,J)*RINV
   10 CONTINUE
C
      J2DOT=0.
      J3DOT=0.
      EUDOT=0.
      EAUDOT=0.
      DO 20 I=1,2
      DO 20 J=1,2
   20 XDOT(I,J)=0.
      RETURN
C
C           LOOP TO COMPUTE X,KX,KXZ,KZ,UOPT,J
 1003 CONTINUE
C
C        SET UP TO COMPUTE
      CALL MMPY(A,X,XDOT1,2,2,1)
      DO 100 I=1,2
  100 XDOT2(I,1)=B(I,1)*UOPT
C
C
C        GET DIZTURBANCES,MISSILE AND TARGET
      WINDM=0.
      IF(T.LT.1.7.OR.T.GT.2.5) GO TO 1001
      WINDM=32.2*SIN(3.927*(T-1.7))*SIN(3.927*(T-1.7))
```

```
 1031 CONTINUE
      WM=WINDM-32.2*CALH
C
C
C         GET COMPOSITE DISTURBANCE
      WT=0.
      W=WM-WT
      DO 101 I=1,2
  101 XDOT3(I,1)=F(I,1)*W
      DO 102 I=1,2
  102 XDOT(I,1)=XDOT1(I,1)+XDOT2(I,1)+XDOT3(I,1)
      Z(1,1)=W
C
C         GET DISTURBANCE DERIVATIVES, MISSILE AND TARGET
      Z(2,1)=WD
      XMR=XMRIC+VMR*T
      XTR=XTRIC+VTR*T
      VMN=VTN+X(2,1)
      XMN=XTN+X(1,1)
C
C
C         SET UP FOR KX,KXZ,KZ FORWARD INTEGRATIONS
      CALL MMPY (KX,A,ARA,2,2,2)
      CALL MMPY(AT,KX,ARB,2,2,2)
      CALL MMPY(KX,BBT,KXBBT,2,2,2)
      CALL MMPY(KXBBT,KX,ARC,2,2,2)
      CALL MMPY(KXBBT,KXZ,ARD,2,2,2)
      CALL MMPY(AT,KXZ,ARE,2,2,2)
      CALL MMPY(KXZ,D,ARF,2,2,2)
      CALL MMPY(KX,FH,ARG,2,2,2)
      CALL MMPY(KZ,D,ARH,2,2,2)
      CALL MMPY(BBT,KXZ,BBTKXZ,2,2,2)
      DO 105 J=1,2
      DO 105 I=1,2
  105 KXZTR(I,J)=KXZ(J,I)
      CALL MMPY(KXZTR,BBTKXZ,ARI,2,2,2)
      CALL MMPY(KXZTR,FH,ARJ,2,2,2)
      DO 106 J=1,2
      DO 106 I=1,2
      ARHT(I,J)=ARH(J,I)
  106 ARJT(I,J)=ARJ(J,I)
C
C
C         GET DERIVATIVES FOR FORWARD INTEGRATION
      DO 110 J=1,2
      DO 110 I=1,2
      KXDOT(I,J)=-ARA(I,J)-ARB(I,J)+ARC(I,J)*RINV-Q(I,J)
      KXZDOT(I,J)=-ARG(I,J)-ARF(I,J)-ARE(I,J)+ARD(I,J)*RINV
      KZDOT(I,J)=-ARJ(I,J)-ARH(I,J)-ARJT(I,J)-ARHT(I,J)+ARI(I,J)*RINV
  110 CONTINUE
C
C
C         COMPUTE UOPT
      CALL MMPY(KX,X,V1,2,2,1)
      CALL MMPY(KXZ,Z,V2,2,2,1)
```

```
      DO 120 I=1,2
  120 V3(I,1)=V1(I,1)+V2(I,1)
      USC=0.0
      DO 121 I=1,2
  121 USC=USC+B(I,1)*V3(I,1)
      UOPT=-RINV*USC
C
C
C        COMPUTE PERFORMANCE INDEX AT T.LE.TF
      CALL MMPY(Q,X,V5,2,2,1)
      DO 122 I=1,2
  122 J2S=J2S+X(I,1)*V5(I,1)
      J2DOT=0.5*J2S
      J3DOT=0.5*UOPT**2/(RINV+1.E-20)
      EUDOT=J3DOT*RINV
      EAUDOT=SQRT(UOPT*UOPT)
      IF (.NOT.LAST) RETURN
C        COMPUTE BURDEN, ASSISTANCE AND UTILIZATION IN RECORD RUN
      CALL MMPY(KZ,Z,V4,2,2,1)
      BSC=0.0
      DO 210 I=1,2
  210 BSC=BSC+Z(I,1)*V4(I,1)
      BURDEN = 0.5*BSC
      CALL MMPY(KXZ,Z,V4,2,2,1)
      ASC=0.0
      DO 230 I=1,2
  230 ASC=ASC+X(I,1)*V4(I,1)
      ASSIST=-ASC
      UTIL=ASSIST-BURDEN
      XT=XTR*CALH+XTN*SALH
      YT=-XTR*SALH+XTN*CALH
      XM=XMR*CALH+XMN*SALH
      YM=XMN*CALH-XMR*SALH
      XMD=VMR*CALH+VMN*SALH
      YMD=VMN*CALH-VMR*SALH
      XMD=XMD+1.E-20
      THMV=ATAN2(YMD,XMD)*57.2958
      THMVP=THMV
      XTP=XT
      YTP=YT
      XMP=XM
      YMP=YM
      KXP(1)=KX(1,1)
      KXP(2)=KX(2,1)
      KXP(3)=KX(1,2)
      KXP(4)=KX(2,2)
      KXZP(1)=KXZ(1,1)
      KXZP(2)=KXZ(2,1)
      KXZP(3)=KXZ(1,2)
      KXZP(4)=KXZ(2,2)
      KZP(1)=KZ(1,1)
      KZP(2)=KZ(2,1)
      KZP(3)=KZ(1,2)
      KZP(4)=KZ(2,2)
      XP(1)=X(1,1)
```

```
      XP(2)=X(2,1)
      ZP(1)=Z(1,1)
      ZP(2)=Z(2,1)
      UOPTP=UOPT
      XTNP=XTN
      XTRP=XTR
      XMNP=XMN
      XMRP=XMR
      END
**TRANSLATION TIME =    0.339***
```

```
SET A=0.0,0.0,1.0,0.0
SET ALM=30.
SET  AT=0.0,1.0,0.0,0.0
SET B=0.,1.
SET BBT=0.,0.,0.,1.
SET C=1.0,0.0
SET CINTFD=0.5
SET CINTBD=0.5
SET D=0.,0.,1.,0.
SET DTBCK=0.02
SET DTFWD=0.02
SET F=0.0,1.0
SET H=1.0,0.0
SET  FH=0.0,1.0,0.0,0.0
SET IALG=5
SET KX=1.0,0.0,0.0,0.0
SET KXZ=0.0,0.0,0.0,0.0
SET KZ=0.0,0.0,0.0,0.0
SET Q=0.0,0.0,0.0,0.0
SET RINV=1.
SET S=50.,0.,0.,10.
SET TFINIT=6.
SET VMR=2000.
SET VTNIC=0.
SET VTR=0.
SET WMXTB=0.,1.0,1.35,1.65,2.4,4.0
SET WMYTB=-27.9,-27.9,-91.0,-91.0,-27.9,-27.9
SET WTXTB=0.0,  1.0,  2.0,  2.1,  3.0,  4.0
SET WTYTB=0.0,  0.0,  0.0,  64.4,  64.4,  64.4
SET WDMXTB = 0.0,0.01,1.0,2.0,2.01,3.0,3.01,3.1,3.11,4.0
SET WDTYTB = 0.0,0.0,644.,644.,0.0,0.0
SET WUTXTB = 0.0,2.0,2.01,2.1,2.11,4.0
SET XIC=300.,0.
SET XMRIC=-9000.
SET XTRIC=0.
SET XTNIC=0.
PREPAR T,XP,ZP,KXP,KXZP,KZP,W,WT,WM,          ...
BURDEN,ASSIST,UTIL,JZERO,TFZERO,UOPTP
PREPAR XTNP,XTRP,XMNP,XMRP
PREPAR THMVP
PREPAR XTP,YTP,XMP,YMP
PREPAR XMD,YMD
PREPAR XML,YML
```

B.4 Program DUCAT3

This section contains a listing and parameter definitions for the DUCAT3 program which solves for a disturbance-utilizing control or a conventional linear-quadratic control in the case of zero set-point problems with a fourth-order plant and fourth-order disturbance model (or two second-order disturbance models). Input parameters determined by "SET" statements, and plot output parameters identified by "PREPAR" statements at the end of the program, are defined below. Printed output parameters are also defined.

INPUT PARAMETERS

PARAMETER	DEFINITION
A	A as defined in Chapter IV.
AT	Transpose of A.
B	B as defined in Chapter IV.
BT	Transpose of B.
CDZXT	Drag coefficient table, independent parameter.
CDZYT	Drag coefficient table, dependent parameter.
CINTBD	Backward-time communication interval (sec).

CINTFD	Forward-time communication interval (sec).
D	D as defined in Chapter IV.
DRAGC	Drag constant, $\frac{1}{2}\rho S_m C_D$ as in Chapter IV.
DTBCK	Backward-time integration interval (sec).
DTFWD	Forward-time integration interval (sec).
F	F as defined in Chapter IV.
FH	FH as defined in Chapter IV.
H	H as defined in Chapter IV.
IALG	Logic input. Selects fourth-order Runge-Kutta integration when IALG = 5.
MMXT	Missile mass table, independent parameter (time).
MMYT	Missile mass table, dependent parameter (slugs).
R	R as defined in Chapter IV.
RINV	Inverse of R.
S	S as defined in Chapter IV.
TFINIT	Maximum value T_{max} for final-time scan (sec).
TMANXT	Target maneuver disturbance table, independent variable.
TMANYT	Target maneuver disturbance table, dependent variable (ft/sec^2).
TWN1YT	Horizontal target wind disturbance table, dependent variable (ft/sec^2).

TWN3YT	Vertical target wind disturbance table, dependent variable (ft/sec^2).
TWN3XT	Vertical target wind disturbance table, independent variable.
VS	Velocity of sound, low altitude (ft/sec).
WND3XT	Vertical missile wind disturbance table, independent variable.
WND3YT	Vertical missile wind disturbance table, dependent variable (ft/sec^2).
XIC	Initial condition of x.
XT1IC	Initial condition of horizontal target position (ft).
XT2IC	Initial condition of horizontal target velocity (ft/sec).
XT3IC	Initial condition of vertical target position (ft).
XT4IC	Initial condition of vertical target velocity (ft/sec).

PLOT OUTPUT PARAMETERS

PARAMETER	DEFINITION
T	Program time (sec).
XP	x as defined in Chapter IV.
ZP	z as defined in Chapter IV.
KXP	K_x as defined in Chapter IV.

KXZP	K_{xz} as defined in Chapter IV.
W1P	w_1 as defined in Chapter IV.
W2P	w_2 as defined in Chapter IV.
UOPTP	u° as defined in Chapter IV.
UTIL	Utility as defined in Chapter IV.
BURDEN	Burden as defined in Chapter IV.
ASSIST	Assistance as defined in Chapter IV.
CDZP	Drag coefficient C_D as in Chapter IV.
MMP	Missile mass (slugs) as in Chapter IV.
DM	Base drag of missile (ft/sec^2).
D1	Horizontal component of DM.
D2	Vertical component of DM.
THMANP	Angle of target maneuver force relative to ground (degrees).
VELTP	Target velocity magnitude (ft/sec).
VELMP	Missile velocity magnitude (ft/sec).
TMANP	Target maneuver acceleration magnitude (ft/sec^2).
ULONGP	Missile control force, longitudinal (pounds).
ULATP	Missile control force, lateral (pounds).
UOPTAP	Missile control force, resultant (pounds).
WIND1	Horizontal missile wind disturbance acceleration (ft/sec^2).

WIND2	Vertical missile wind disturbance acceleration (ft/sec^2).
WNDT1P	Horizontal target wind disturbance acceleration (ft/sec^2).
WNDT2P	Vertical target wind disturbance acceleration (ft/sec^2).
XM1P	Horizontal missile position (ft).
XM3P	Vertical missile position (ft).
XT1P	Horizontal target position (ft).
XT3P	Vertical target position (ft).
THMP	Missile velocity vector angle (degrees).
THTP	Target velocity vector angle (degrees).
XT2P	Horizontal target velocity (ft/sec).
XT4P	Vertical target velocity (ft/sec).
XM2P	Horizontal missile velocity (ft/sec).
XM4P	Vertical missile velocity (ft/sec).
XT2DTP	Horizontal target acceleration (ft/sec^2).
XT4DTP	Vertical target acceleration (ft/sec^2).

PRINTED OUTPUT PARAMETERS

PARAMETER	DEFINITION
KXT	Stored initial-condition of K_x.
KXZT	Stored initial-condition of K_{xz}.
KZT	Stored initial-condition of K_z.
TAUTAB	A particular specified terminal time T_i.
JMATAB	Value of $\frac{1}{2}\, x^T(T)\, Sx(T)$
J2TAB	Value of $\frac{1}{2} \int_0^T x^T(t)\, Qx(t)\, dt$
J3TAB	Value of $\frac{1}{2} \int_0^T u^T(t)\, Ru(t)\, dt$
JTFTAB	Value of J at $t = T$.
X1TAB	$x_1(T)$.
X3TAB	$x_3(T)$.
EAUTAB	EAU as defined in Chapter IV.
EUTAB	EU as defined in Chapter IV.

The listing of Program DUCAT3 follows:

DUCAT3

```
PROGRAM OPTG
COMMENT   COMPUTE OPTIMAL CONTROL FOR CASE OF MISSILE INTERCEPT ...
C
INITIAL
      ARRAY     A(4,4),ARA(4,4),ARB(4,4),ARC(4,4),ARD(4,4),ARE(4,4)
      ARRAY     ARG1(4,4)
      ARRAY     ARF(4,4),ARG(4,4),ARH(4,4),ARHT(4,4),ARI(4,4),ARJ(4,4)
      ARRAY     ARI1(4,4)
      ARRAY     ARJT(4,4),ARK(4,4),AT(4,4),B(4,2),BRINV(4,2),BT(2,4)
      ARRAY CDZXT(7),CDZYT(7)
      ARRAY     D(4,4),DMXT(4),DMYT(4),F(4,2),FH(4,4),H(2,4)
      ARRAY     JTFTAB(10),KX(4,4),KXIC(4,4),KXP(4,4),KXT(4,4,10)
      ARRAY JMATAB(10),J2TAB(10),J3TAB(10),LAUTAB(10),EUTAB(10)
      ARRAY     KXDOT(4,4)
      ARRAY     KXZ(4,4),KXZIC(4,4),KXZP(4,4),KXZT(4,4,10),KXZTR(4,4)
      ARRAY     KXZDOT(4,4)
      ARRAY KZ(4,4)
      ARRAY     KZDOT(4,4)
      ARRAY     KZIC(4,4),KZP(4,4),KZT(4,4,10),Q(4,4),QX(4,1),R(2,2)
      ARRAY MMXT(5),MMYT(5)
      ARRAY     RINV(2,2),RINVBT(2,4),PUOPT(2,1),S(4,4),SX(4,1)
      ARRAY TMANXT(4),TMANYT(4)
      ARRAY TAUTAB(10)
      ARRAY TWN1XT(6),TWN1YT(6),TWN3XT(6),TWN3YT(6)
      ARRAY     UOPT(2,1),UOPTP(2,1),V1(4,1),V2(4,1),V3(4,1),V4(4,1)
      ARRAY V5(4,1)
      ARRAY WND3XT(6),WND3YT(6)
      ARRAY     X(4,1),XIC(4,1),XP(4,1),X1TAB(10),X3TAB(10)
      ARRAY     XDOT(4,1)
      ARRAY     XDOT1(4,1),XDOT2(4,1),XDOT3(4,1)
      ARRAY     Z(4,1),ZP(4,1)

      INTEGER I,J
      INTEGER NS
      INTEGER NSFNL
      INTEGER NERR1,NERR2,NERR3,NERR4,NERR5 ,NERR6,NERR7,NERR8,NERR9

      CONSTANT A=16*0.,AT=16*0.,B=8*0.,BT=8*0.,D=16*0.
      CONSTANT CDZXT=7*0.,CDZYT=7*0.
      CONSTANT CDZP=0.,MMP=0.
      CONSTANT DMXT=4*0.,DMYT=4*0.
      CONSTANT DM=0.,D1=0.,D2=0.
      CONSTANT DRAGC=0.
      CONSTANT DTBCK=0.,DTFWD=0.
      CONSTANT F=8*0.,FH=16*0.,GRAV=32.2,H=8*0.
      CONSTANT KX=16*0.,KXZ=16*0.,KZ=16*0.
      CONSTANT MACHP=0.
      CONSTANT VELMP=0.
      CONSTANT MMXT=5*0.,MMYT=5*0.
      CONSTANT Q=16*0.,R=4*0.,RINV=4*0.,S=16*0.
      CONSTANT TFINIT=2.0
      CONSTANT TMANP=0.,THMANP=0.,W1P=0.,W2P=0.,VELTP=0.
      CONSTANT TMAN=0.,TMANXT=4*0.,TMANYT=4*0.
```

```
      CONSTANT TWN1XT=6*0.,TWN1YT=6*0.
      CONSTANT TWN3XT=6*0.,TWN3YT=6*0.
      CONSTANT UOPTAP=0.
      CONSTANT ULONGP=0.,ULATP= .
      CONSTANT V0=0.
      CONSTANT WD1=0.,WD2=0.
      CONSTANT WIND1=0.,WIND2=0.,W1=0.,W2=0.
      CONSTANT WND3XT=6*0.,WND3YT=6*0.
      CONSTANT WNDT1P=0.,WNDT2P=0.
      CONSTANT XIC=4*0.
      CONSTANT XT1IC=0.,XT3IC=0.,XT2IC=0.,XT4IC=0.
      CONSTANT XT2DOT=0.,XT4DOT=0.
      CONSTANT XT2DTP=0.,XT4DTP=0.
      CONSTANT XT2P=0.,XT4P=0.,XM2P=0.,XM4P=0.
      CONSTANT XM1P=0.,XM3P=0.,XT1P=0.,XT3P=0.,THMP=0.,THTP=0.


      LOGICAL FWDBWD,LAST


      NS=0
      DO BWDIC I=1,4
      DO BWDIC J=1,4
      KXIC(I,J)=S(I,J)
      KXZIC(I,J)=0.
      KZIC(I,J)=0.
BWDIC..CONTINUE
      FWDBWD=.FALSE.
      LAST=.FALSE.
LOOP..CONTINUE
      CONSTANT CINTLF=0.02
      CONSTANT CINTFD=0.1,CINTBD=0.1
      MAXT=RSW(FWDBWD,DTFWD,DTBCK)
      CONSTANT UOPT=2*0.
      BURDEN = 0.0
      ASSIST = 0.0
      UTIL = 0.0
      CONSTANT X=4*0.
      CONSTANT Z=4*0.
      TSTP = 130.
      CONSTANT KXP=16*0.,KXZP=16*0.,KZP=16*0.
      CONSTANT XP=4*0.,ZP=4*0.
      CONSTANT UOPTP=2*0.
      J2=0.
      J3=0.
      EAU=0.
      EU=0.
      J2S=0.
      JMAYER=0.
      TL=0.
      MACH=0.
      MM=0.
      W1=0.
      W2=0.
      W1L=0.
```

```
      W2L=0.
      W3I=0.
      WD2=0.
      XT1=XT1IC
      XT2=XT2IC
      XT3=XT3IC
      XT4=XT4IC
      XM1=XIC(1,1)+XT1
      XM2=XIC(2,1)+XT2
      XM3=XIC(3,1)+XT3
      XM4=XIC(4,1)+XT4
END  $  "INITIAL"
DYNAMIC
      CINTERVAL  CINT=0.1
      MAXTERVAL  MAXT=0.0.1
      NSTEPS  NSTP=1
      CINTFL=RSW(LAST,CINTLF,CINTFD)
      CINT=RSW(FWDBWD,CINTFL,CINTBD)
      VARIABLE  T=0.
ALGORITHM  IALG=4
      IF (FWDBWD) GO TO SKPSV
      IF(T.LE.0.) GO TO SKPSV
      NS=NS+1
      DO SVLOOP I=1,4
      DO SVLOOP J=1,4
      KXT(I,J,NS)=KX(I,J)
      KXZT(I,J,NS)=KXZ(I,J)
      KZT(I,J,NS)=KZ(I,J)
SVLOOP..CONTINUE
      GO TO SKPDER
SKPSV..CONTINUE
      WD1=RSW(T.EQ.0.,0.,(W1-W1L)/(T-TL))
      WD2=RSW(T.EQ.0.,0.,(W2-W2L)/(T-TL))
      W1L=W1
      W2L=W2
      TL=T
SKPDER..CONTINUE
      TERMT(ABS(T-TFINIT).LE.CINTBD.AND..NOT.FWDBWD)
      IF(FWDBWD) TSTP=CINTBD*FLOAT(NS)
      IF(FWDBWD) CINT=AMIN1(CINTFL,TSTP-T)
      CONSTANT TGOMN=1.E-5
TERMT(ABS(T-TSTP).LE.TGOMN.AND.FWDBWD)
DERIVATIVE  OPTGN
      CALL FTNDRV
      KX=INTVC(KXDOT,KXIC)
      KXZ=INTVC(KXZDOT,KXZIC)
      KZ=INTVC(KZDOT,KZIC)
      X=INTVC(XDOT,XIC)
      J2=INTEG(J2DOT,0.)
      J3=INTEG(J3DOT,0.)
      EAU=INTEG(EAUDOT,0.)
      EU=INTEG(EUDOT,0.)
      XT2=INTEG(XT2DOT,XT2IC)
      XT4=INTEG(XT4DOT,XT4IC)
      XT1=INTEG(XT2,XT1IC)
```

```
      XT3=INTEG(XT4,XT3IC)
      XM1=X(1,1)+XT1
      XM3=X(3,1)+XT3
      XM2=X(2,1)+XT2
      XM4=X(4,1)+XT4
END  $  "DERIVATIVE"
END  $  "DYNAMIC"
TERMINAL
      IF (FWDBWD.AND.LAST) GO TO TERM
      IF (FWDBWD) GO TO CYCLE
      WRITE (6,98) (((KXT(II,JJ,KK),KXZT(II,JJ,KK),    ...
           KZT(II,JJ,KK),II=1,4),JJ=1,4),KK=1,NS)
98..  FORMAT(T20,3E15.5)
      NSFNL=NS
      NS=1
      FWDBWD=.TRUE.
      GO TO SETIC
CYCLE..CONTINUE
      CALL MMPY(S,X,SX,4,4,1)
      SCM=0.0
      DO JT2 I=1,4
JT2.. SCM=SCM+X(I,1)*SX(I,1)
      JMAYER=0.5*SCM
      JTF=JMAYER+J2+J3
      JMATAB(NS)=JMAYER
      J2TAB(NS)=J2
      J3TAB(NS)=J3
      EAUTAB(NS)=EAU
      EUTAB(NS)=EU
      X1TAB(NS)=X(1,1)
      X3TAB(NS)=X(3,1)
      JTFTAB(NS)=JTF
      TAUTAB(NS) = T

SETIC..NS=NS+1
      IF (NS.GT.NSFNL) GO TO FNLRUN
RECUR.. DO IC1 J=1,4
      DO IC1 I=1,4
      KXIC(I,J)=KXT(I,J,NS)
      KXZIC(I,J)=KXZT(I,J,NS)
      KZIC(I,J)=KZT(I,J,NS)
IC1..CONTINUE
LOG

      GO TO LOOP
FNLRUN..CONTINUE
      JZERO=1.E30
      DO JMIN I=1,NSFNL
      IF (JZERO.LT.JTFTAB(I)) GO TO JMIN
      JZERO=JTFTAB(I)
      TFZERO = TAUTAB(I)
      NS=I
JMIN..CONTINUE
      NS=6
      WRITE(6,299) (TAUTAB(I),JMATAB(I),J2TAB(I),J3TAB(I),  ...
```

```
       JIFTAB(I),X1TAB(I),X3TAB(I),LAUTAB(I),EUTAB(I),I=1,NSFNL)
299.. FORMAT(T4,F4.2,T12,E11.4,T26,E11.4,T40,E11.4,...
       T54,E11.4,T68,E11.4,T82,E11.4,T96,E11.4,T110,E13.4)
      REWIND 6
      LAST=.TRUE.
      GO TO RECOR
TERM..CONTINUE
END  $  "TERMINAL"
END  $  "PROGRAM"
```

```
      SUBROUTINE FINDRV
$
      IF (FWDBWD) GO TO 1003
      DO 5 J=1,4
      DO 5 I=1,4
    5 KXZTR(I,J)=KXZ(J,I)
      CALL MMPY (KX,A,ARA,4,4,4)
      CALL MMPY (AT,KX,ARB,4,4,4)
      CALL MMPY(B,RINV,BRINV,4,2,2)
      CALL MMPY(BRINV,BT,ARK,4,2,4)
      CALL MMPY(KX,ARK,ARC1,4,4,4)
      DO 4 J=1,4
      DO 4 I=1,4
    4 ARD(I,J)=ARC1(I,J)
      CALL MMPY(ARC1,KX,ARC,4,4,4)
      CALL MMPY(ARD,KXZ,ARD,4,4,4)
      CALL MMPY(KXZTR,ARK,ARI1,4,4,4)
      CALL MMPY(ARI1,KXZ,ARI,4,4,4)
      CALL MMPY(KXZTR,FH,ARJ,4,4,4)
      CALL MMPY (AT,KXZ,ARE,4,4,4)
      CALL MMPY (KXZ,D,ARF,4,4,4)
      CALL MMPY (KX,FH,ARG,4,4,4)
      CALL MMPY (KZ,D,ARH,4,4,4)
      DO 6 J=1,4
      DO 6 I=1,4
      ARHT(I,J)=ARH(J,I)
    6 ARJT(I,J)=ARJ(J,I)
C
C
C        GET DERIVATIVES FOR BACKWARD INTEGRATIONS
      DO 10 J=1,4
      DO 10 I=1,4
      KXDOT(I,J)=ARA(I,J)+ARB(I,J)-ARC(I,J)          +Q(I,J)
      KXZDOT(I,J)=ARG(I,J)+ARF(I,J)+ARE(I,J)-ARD(I,J)
      KZDOT(I,J) =ARJ(I,J)+ARH(I,J)+ARJT(I,J)+ARHT(I,J)-ARI(I,J)
   10 CONTINUE
C
      J2DOT=0.
      J3DOT=0.
      EAUDOT=0.
      EUDOT=0.
      DO 20 I=1,4
   20 XDOT(I,1)=0.
      RETURN
C
C          LOOP TO COMPUTE X,KX,KXZ,KZ,UOPT,J
 1003 CONTINUE
C
C        SET UP TO COMPUTE
      CALL MMPY(A,X,XDOT1,4,4,1)
      CALL MMPY(B,UOPT,XDOT2,4,2,1)
C
C
C        GET DISTURBANCES, MISSILE AND TARGET
```

```
      TTAB=T/(TFINIT+1.E-20)
      VELM=SQRT(XM2*XM2+XM4*XM4)+1.E-20
      MACH=VELM/VS
      CALL INTERP(MACH,CDZXT,CDZYT,7,2,CDZ,NERR6)
      CALL INTERP(TTAB,MMXT,MMYT,6,2,MM,NERR7)
      DM=DRAGC*VELM*VELM*CDZ/MM
      D1=DM*XM2/VELM
      D1=-D1
      D2=DM*XM4/VELM
      D2=-D2
      IF(T.LE.1.1)TMAN=0.
      IF(T.GT.1.1.AND.T.LE.1.6)TMAN=256.*T-281.6
      IF(T.GT.1.6)TMAN=128.
      XT2=XT2+1.E-20
      THT=ATAN2(XT4,XT2)
      TMAN=TMAN+1.E-20
      THMAN=THT+1.5708*(TMAN/SQRT(TMAN*TMAN))
      CALL INTERP(TTAB,TWN1XT,TWN1YT,6,2,WNDT1,NERR8)
      CALL INTERP(TTAB,TWN3XT,TWN3YT,6,2,WNDT2,NERR9)
      WNDT2=0.
      IF(T.LT.1.7.OR.T.GT.2.5) GO TO 101
      WINDM= 32.2*SIN(3.927*(T-1.7))*SIN(3.927*(T-1.7))
      WIND1=WINDM*XM4/VELM
      WIND2=WINDM*XM2/VELM
      WIND1=-WIND1
  101 CONTINUE
      XT2DOT=WNDT1+SQRT(TMAN*TMAN)*COS(THMAN)
      XT4DOT=WNDT2+SQRT(TMAN*TMAN)*SIN(THMAN)
      W1=WIND1+D1-XT2DOT
      W2=WIND2+D2-GRAV-XT4DOT
C        GET DISTURBANCE DERIVATIVES
      Z(1,1)=W1
      Z(2,1)=WD1
      Z(3,1)=W2
      Z(4,1)=WD2
C
C
C        GET COMPOSITE DISTURBANCE
      CALL MMPY(FM,Z,XDOT3,4,4,1)
      DO 102 I=1,4
  102 XDOT(I,1)=XDOT1(I,1)+XDOT2(I,1)+XDOT3(I,1)
C
C
C
C        SET UP FOR KX,KXZ,KZ FORWARD INTEGRATIONS
      DO 105 J=1,4
      DO 105 I=1,4
  105 KXZTR(I,J)=KXZ(J,I)
      CALL MMPY (KX,A,ARA,4,4,4)
      CALL MMPY (AT,KX,ARB,4,4,4)
      CALL MMPY(B,RINV,BRINV,4,2,2)
      CALL MMPY(BRINV,BT,ARK,4,2,4)
      CALL MMPY(KX,ARK,ARC1,4,4,4)
      DO 104 J=1,4
      DO 104 I=1,4
```

```
  104 ARD(I,J)=ARD1(I,J)
      CALL MMPY(ARC1,KX,ARC,4,4,4)
      CALL MMPY(ARD,KXZ,ARD,4,4,4)
      CALL MMPY(KXZTR,ARK,ARI1,4,4,4)
      CALL MMPY(ARI1,KXZ,ARI,4,4,4)
      CALL MMPY(KXZTR,FH,ARJ,4,4,4)
      CALL MMPY (AT,KXZ,ARE,4,4,4)
      CALL MMPY (KXZ,D,ARF,4,4,4)
      CALL MMPY (KX,FH,ARG,4,4,4)
      CALL MMPY (KZ,D,ARH,4,4,4)
      DO 106 J=1,4
      DO 106 I=1,4
      ARHT(I,J)=ARH(J,I)
  106 ARJT(I,J)=ARJ(J,I)
C
C
C        GET DERIVATIVES FOR FORWARD INTEGRATION
      DO 110 J=1,4
      DO 110 I=1,4
      KXDOT(I,J)=-ARA(I,J)-ARB(I,J)+ARC(I,J)      -Q(I,J)
      KXZDOT(I,J)=-ARG(I,J)-ARF(I,J)-ARE(I,J)+ARD(I,J)
      KZDOT(I,J)=-ARJ(I,J)-ARH(I,J)-ARJT(I,J)-ARHT(I,J)+ARI(I,J)
  110 CONTINUE
C
C
C        COMPUTE UOPT
      CALL MMPY(KX,X,V1,4,4,1)
      CALL MMPY(KXZ,Z,V2,4,4,1)
      DO 120 I=1,4
  120 V3(I,1)=V1(I,1)+V2(I,1)
      CALL MMPY(RINV,BT,RINVBT,2,2,4)
      CALL MMPY(RINVBT,V3,UOPT,2,4,1)
      UOPT(1,1)=-UOPT(1,1)
      UOPT(2,1)=-UOPT(2,1)
C
C
C        COMPUTE PERFORMANCE INDEX AT T.LE.TF
      CALL MMPY(R,UOPT,RUOPT,2,2,1)
      VDOT1=0.
      DO 205 I=1,2
  205 VDOT1=VDOT1+UOPT(I,1)*RUOPT(I,1)
      J3DOT=0.5*VDOT1
      CALL MMPY(Q,X,QX,4,4,1)
      VDOT2=0.
      DO 206 I=1,4
  206 VDOT2=VDOT2+X(I,1)*QX(I,1)
      J2DOT=0.5*VDOT2
      VDOT3=0.
      DO 207 I=1,2
  207 VDOT3=VDOT3+UOPT(I,1)*UOPT(I,1)
      EUDOT=0.5*VDOT3
      XM2=XM2+1.E-20
      THM=ATAN2(XM4,XM2)
      THMP=THM*57.2958
      ULONG=UOPT(1,1)*COS(THM)+UOPT(2,1)*SIN(THM)
```

```
      ULAT=UOPT(2,1)*COS(THM)-UOPT(1,1)*SIN(THM)
      EAUDOT=0.5*(SQRT(ULONG*ULONG)+SQRT(ULAT*ULAT))
      IF (.NOT.LAST) RETURN
C        COMPUTE BURDEN, ASSISTANCE AND UTILIZATION IN RECORD RUN
      CALL MMPY(KZ,Z,V4,4,4,1)
      BSC=0.0
      DO 210 I=1,4
  210 BSC=BSC+Z(I,1)*V4(I,1)
      BURDEN = 0.5*BSC
      CALL MMPY(KXZ,Z,V4,4,4,1)
      ASC=0.0
      DO 230 I=1,4
  230 ASC=ASC+X(I,1)*V4(I,1)
      ASSIST=-ASC
      UTIL=ASSIST-BURDEN
      DO 232 I=1,4
      DO 232 J=1,4
  232 KXP(I,J)=KX(I,J)
      KXZP(I,J)=KXZ(I,J)
      KZP(I,J)=KZ(I,J)
      DO 234 I=1,4
      DO 234 J=1,4
      ZP(I,1)=Z(I,1)
  234 XP(I,1)=X(I,1)
      UOPTP(1,1)=UOPT(1,1)
      UOPTP(1,1)=UOPTP(1,1)*MM
      UOPTP(2,1)=UOPT(2,1)
      UOPTP(2,1)=UOPTP(2,1)*MM
      UOPTAP=SQRT(UOPT(1,1)*UOPT(1,1)+UOPT(2,1)*UOPT(2,1))
      UOPTAP=UOPTAP*MM
      ULONGP=ULONG*MM
      ULATP=ULAT*MM
      XT1P=XT1
      XT3P=XT3
      THTP=THT*57.2958
      XM1P=XM1
      XM3P=XM3
      WNOT1P=WNOT1
      WNOT2P=WNOT2
      CDZP=CDZ
      MMP=MM
      VELMP=VELM
      MACHP=MACH
      THMANP=THMAN*57.2958
      TMANP=TMAN
      VELTP=SQRT(XT2*XT2+XT4*XT4)
      XT2P=XT2
      XT4P=XT4
      XM2P=XM2
      XM4P=XM4
      W1P=W1
      W2P=W2
      XT2DTP=XT2DOT
      XT4DTP=XT4DOT
      END

**TRANSLATION TIME =    7.239***
```

```
SET A=        0.,0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,      ...
              0.,0.,0.,0.,0.,0.,0.,0.,1.,0.,0.
SET AT=       0.,1.,0.,0.,0.,0.,0.,0.,0.,0.,0.,      ...
              0.,0.,0.,0.,1.,0.,0.,0.,0.,0.,0.
SET B=        0.,1., 0.,0.,0.,0.,0.,0.,1.
SET BT=       0.,0.,0.,0.,0.,0.,0.,0.,0.,1.
SET CDZXT=0.,0.7,0.95,1.05,1.2,1.6,3.0
SET CDZYT=0.180,0.143,0.285,0.454,0.437,0.328,0.231
SET CINTBD=0.5
SET CINTFD=0.5
SET D=        0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,      ...
              0.,0.,0.,0.,0.,0.,1.,0.,0.
SET DRAGC=0.002705
SET DTBCK=0.02
SET DTFWD=0.02
SET F=        0.,1.,0.,0.,0.,0.,0.,0.,1.0
SET FM=       0.,1.,0.,0.,0.,0.,0.,0.,0.,      ...
              0.,0., 0.,1.,0.,0.,0.,0.,0.
SET H=        1.,0.,0.,0.,0.,0.,1.,0.,0.,0.
SET IALG=5
SET MMXT=0.,0.25,0.5,0.75,1.0
SET MMYT=101.4,95.5,85.5,81.0,73.5
SET R=1.,0.,0.,1.
SET RINV=1.,0.,0.,1.
SET S=100.,0.,0.,0.,0.,0.,0.,0.,      ...
0.,0.,100.,0.,0.,0.,0.,0.
SET TFINIT=5.
SET TMANXT=0.,0.5,0.6,1.
SET TMANYT=0.,0.,-128.,-128.
SET TWN1YT=0.,0.,0.,0.,0.,0.
SET TWN1XT=0.,0.1,0.2,0.3,0.5,1.0
SET TWN3YT=0.,0.,-32.,-32.,0.,0.
SET TWN3XT=0.,0.5,0.6,0.65,0.75,1.0
SET VS=1098.
SET WND3XT=0.0,0.5,0.6,0.65,0.75,1.0
SET WND3YT=0.,0.,0.,0.,0.,0.
SET XIC=-9000.,3000.,1000.,0.
SET XT1IC=0.
SET XT2IC=-1000.
SET XT3IC=4000.
SET XT4IC=0.
PREPAR T,XP,ZP,KXP,KXZP,KZP,W1P,W2P,UOPTP,UTIL,BURDEN,ASSIST
PREPAR CDZP,MMP
PREPAR CM,D1,D2
PREPAR THMANP,VELTP,VELMP,TMANP
PREPAR ULONGP,ULATP
PREPAR UOPTAP
PREPAR WIND1,WIND2
PREPAR WNDT1P,WNDT2P
PREPAR XM1P,XM3P,XT1P,XT3P,THMP,THTP
PREPAR XT2P,XT4P,XM2P,XM4P
PREPAR XT2DTP,XT4DTP
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